A FIXED POINT THEOREM FOR MULTI-VALUED FUNCTIONS

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Let K be an n-cell in Euclidean n-space ($n \geq 2$), and let $T: K \to K$ be a multi-valued function such that for each $x \in K$, $T(x)$ is the boundary $(n - 1)$-sphere of an n-cell in K. Then [2] asserts that T has a fixed point if either (1) T is continuous, or (2) T is upper semi-continuous and there is an $\epsilon > 0$ such that for each $x \in K$, the interior of $T(x)$ contains an ϵ-neighborhood in R^n. The following example shows that the first of these assertions is incorrect. The second assertion, though deduced in [2] from the first, is correct, being a corollary of the theorem proved below.

EXAMPLE. *Let K be the subspace $\{t \mid \| t \| \leq 1\}$ of R^n. If $x \in K$, let $T(x)$ be $\{t \mid \| t - x \| = \rho \;\&\; \| t \| \leq 1\} \cup \{t \mid \| t - x \| \geq \rho \;\&\; \| t \| = 1\}$, where $\rho = 1 - \| x \| + \| x \|^2$.*

Using reduced Čech homology theory with a field of coefficients, we consider a strengthened form of upper semi-continuity for an arbitrary multi-valued function $F: X \to Y$. (3^k) If $x \in X$ and U is a neighborhood of $F(x)$, there is a neighborhood V of x such that if $x' \in V$ then $F(x') \subset U$ and each k-cycle on $F(x)$ is homologous in U to a cycle on $F(x')$.

LEMMA. *For T as above, (2) implies (3^{n-1}).*

Proof. If $x \in K$ and U is a neighborhood of $T(x)$ in K, then since $T(x)$ is an $(n - 1)$-sphere, there exist neighborhoods $W \subset V \subset U$ such that V does not contain an ϵ-neighborhood in R^n [ϵ as in (2)] and every cycle on W is homologous in V to a cycle on $T(x)$. Let N be a neighborhood of x such that if $x' \in N$, then $T(x') \subset W$. If z' is a non-zero $(n - 1)$-cycle on $T(x')$, let z be a cycle on $T(x)$ homologous in V to z'. But z' links [3] an ϵ-neighborhood in R^n, hence links a point of $K - V$, and thus does not bound in V. Consequently, z does not bound in the $(n - 1)$-sphere $T(x)$, so (3^{n-1}) holds.

The converse of this remark may be established in a similar way, and without any restriction on the values $T(x)$ of the function T. However, the following theorem does not hold if (3^{n-1}) is replaced by (2).

THEOREM. *Let X be a compact, homologically trivial ANR in Euclidean space R^n. A multi-valued function $F: X \to X$ has a fixed point provided that (3^{n-1}) holds and that, if $x \in X$ and $0 \leq q \leq n - 2$, then $H_q(F(x)) = 0$.*

Proof. If $x \in X$, let $G(x) = R^n - D(x)$, where $D(x)$ is the unbounded component of $R^n - F(x)$. Since $R^n - X$ is connected, $G(x) \subset X$. By the Alexander duality theorem [3], the infinite homology group $h_q^a(R^n - F(x))$ vanishes if $q > 0$. Hence $h_q^a(D(x))$ vanishes if $q \geq 0$, so $G(x)$ is homologically trivial. If $y \in X - G(x)$, since F is upper semi-continuous, there is a neighborhood V of x in X such