THE EXISTENCE OF INDEFINITE TERNARY GENERA OF MORE THAN ONE CLASS

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1. Introduction. Consider an indefinite ternary quadratic form

$$f = \sum_{i,j=1}^{3} a_{ij}x_ix_j \qquad (a_{ij} = a_{ji}),$$

whose matrix $A = [a_{ij}]$ is integral. We assume that f is primitive, i.e., that the g.c.d. of all the elements of A is 1. f is called properly primitive if at least one of a_{11}, a_{22}, a_{33} is odd. Otherwise it is called improperly primitive. We set $\tau(f) = 1$ or 2 according as f is properly or improperly primitive.

Let Ω be the g.c.d. of all the 2-rowed minor determinants of A, and let the integer Δ be defined by $|A| = \Omega^2\Delta$. The form F with matrix $\Omega^{-1} | A | A^I$ is called the reciprocal form of f. The g.c.d. of all the 2-rowed minor determinants of F is Δ, and the determinant of this matrix is $\Delta^2\Omega$. Two forms are said to be in the same order if they have the same index and the same numerical invariants $\tau(f)$, $\tau(F)$, Ω, and Δ. If f is indefinite, the index of f is determined by the sign of Δ. Therefore, since we shall be concerned only with indefinite forms, we may denote an order of forms by $\{\tau(f), \tau(F), \Omega, \Delta\}$.

In this paper we shall show that certain orders of forms contain genera having more than one class. Some results of this type are contained in the paper of B. W. Jones and E. H. Hadlock [4]. In §2 we shall employ their method to obtain further orders with the desired property. In §3 we employ the idea of N-related form to extend the results of §2. The N-related forms generalize the p-related forms used by Jones in [3]. In §4 we give an example and discuss the connection of our results with certain historical aspects of the problem.

2. The method of Jones and Hadlock.

LEMMA 1. *Let $\Omega = 2^{\omega}\Omega_1^2\Omega_2$ and $\Delta = -2^{\delta}\Delta_1^2\Delta_2$, where Ω_1, Ω_2, Δ_1, Δ_2 are odd, and Ω_2, $\Delta_2 > 0$. Let p and q be distinct odd primes and prime to $\Omega\Delta$. Assume $p \equiv 1 \pmod 4$ and, if δ is odd, $q \equiv \pm 1 \pmod 8$. Then, if $(\Omega_2 \mid p) = (\Delta_2 \mid p)$, if $(\Delta_2 \mid q) = 1$, and if ω and δ are either both even or both odd, the order $\{1, 1, \Omega, \Delta\}$ contains a form*

$$f = ax_1^2 - \Omega q^2x_2^2 + cx_3^2 + 2r'\Omega x_2x_3 + 2sx_1x_3,$$

where $a = 1$ or p^2, for some values of c, r', and s.

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