THE NORMALITY OF THE PRODUCT OF TWO ORDERED SPACES

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A simply ordered set with its natural topology will be called an *ordered space;* it is well known (see, e.g., [1; 39 and 41]) that every such space is a normal Hausdorff space and is compact if and only if it is complete (i.e., has no *gaps*, as defined below). The present paper consists primarily of two theorems, the first of which shows that for every non-paracompact (see [3]) ordered space X there exists a compact ordered space Y such that $X \times Y$ is not normal. The second theorem gives a necessary and sufficient condition for the normality of $X \times Y$, where X is a locally compact ordered space and Y is a compact ordered space.

The proof of the first theorem makes use of a characterization of paracompact ordered spaces due to Gillman and Henriksen [4, Theorem 9.5]; it will be convenient to repeat some of their definitions before proceeding. (These definitions are given in more detail on p. 347 and p. 353 of [4].)

An *interior gap* of an ordered space X is a Dedekind cut $(A \mid B)$ of X such that A has no last element and B has no first element; such a gap is regarded as a "virtual" element satisfying the expected ordering relations. In case X has no first element, a virtual element u such that $u < x$ for all $x \in X$ is introduced and is referred to as the *left end-gap* of X; if X has no last element, the *right end-gap* of X is defined analogously. The (compact) ordered space consisting of X together with all of its gaps is denoted by X^+.

A gap u of an ordered space X is called a *Q-gap from the left* (*right*) provided there exist a regular initial ordinal ω_α and an increasing (decreasing) sequence $\{x_\xi\}_{\xi<\omega_\alpha}$ of points of X^+ having the limit u such that for every limit ordinal $\lambda < \omega_\alpha$ the limit in X^+ of $\{x_\xi\}_{\xi<\lambda}$ is a gap of X; a gap u of X is called a *Q-gap* if it is a Q-gap from the left and from the right (or only the appropriate one, in case u is an end-gap). The characterization referred to above is that *an ordered space X is paracompact if and only if every gap of X is a Q-gap.*

It is well-known that if X is the space of all ordinals $< \omega_1$ and Y is the space of all ordinals $\leq \omega_1$, then $X \times Y$ is not normal (see, e.g., [2; 68, Exercise 13]). The following theorem is a generalization of this result.

Theorem 1. *If X is a non-paracompact ordered space, then $X \times X^+$ is not normal.*

Proof. Since X is not paracompact, there is a gap u of X which is not a Q-gap. Suppose u is not a Q-gap from the left; i.e., if $\{x_\xi\}_{\xi<\omega_\alpha}$ is an increasing sequence of points of X having the limit u, there is a point x of X and a limit ordinal $\lambda < \omega_\alpha$ such that the sequence $\{x_\xi\}_{\xi<\lambda}$ has the limit x. Let $S = X \times X^+$,

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