THE CHARACTERISTIC ROOTS OF A MATRIX: A CORRECTION

By T. E. Easterfield

In 1952, Sen-Ming Leng [1] published in this Journal a number of inequalities for the characteristic roots of a matrix. Certain of these were based on Hölder's inequality, viz: for a_i, b_i nonnegative and $p > 1$,

$$\sum_i a_i b_i \le (\sum_i a_i^p)^{1/p} \cdot (\sum_j b_j^{p'})^{1/p'}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

The conditions for equality to hold are that the ratio $a_i^p : b_i^{p'}$ shall be independent of i. Leng unfortunately gives the condition as being that $a_i : b_i$ shall be independent of i. Possibly this statement of the condition is taken from Hardy's "Pure Mathematics" (Tenth Edition 1952, p. 489), where, in fact, the error arises through a change of notation in the course of the proof. If the correct condition is used, Leng's Theorem 5 breaks down, as we shall show, with a counterexample. The conditions for the inequalities of Theorems 8 and 9 are also wrongly given.

Leng's Theorem 5 states that, for any $n \times n$ matrix (a_{ij}), if $\lambda = \sum_{ij} u_i a_{ij} \bar{u}_j$, where $\sum_i u_i \bar{u}_i = 1$,

$$|\lambda| \le \max_i (\sum_j | r_{ij} a_{ij} |^{\frac{1}{2}p})^{1/p} \cdot \max_j (\sum_i | r'_{ij} a_{ij} |^{\frac{1}{2}p'})^{1/p'},$$

where $r_{ij} > 0$, $r'_{ij} = 1/r_{ij}$, $p > 1$, $1/p + 1/p' = 1$. Conditions are given for equality to hold.

In fact, the proof breaks down in the step at the top of p. 148, where the wrongly stated condition for equality is used to derive

$$(\sum_i | u_i |^p)^{1/p} \cdot (\sum_i | u_i |^{p'})^{1/p'} = \sum_i | u_i | \cdot | u_i |;$$

which is not true.

Thus, for example, if in the theorem we set

$$A = \begin{pmatrix} 1 & 4 \\ 16 & 64 \end{pmatrix}$$

$$r = \begin{pmatrix} 1 & 4 \\ \frac{1}{16} & \frac{1}{4} \end{pmatrix} \qquad r' = \begin{pmatrix} 1 & \frac{1}{4} \\ 16 & 4 \end{pmatrix}$$

$$u = (\tfrac{1}{65}, \tfrac{8}{65}) \qquad p = 3 \qquad p' = \tfrac{3}{2},$$

then $\bar{u}' Au = (129 \times 33)/65 = 65.49$,

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