AN EXTENSION OF RAMANUJAN'S SUM. III. CONNECTIONS WITH TOTIENT FUNCTIONS

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1. **Introduction.** In this paper we conclude a study of the characteristic properties of the function $c_k(n, r)$, defined for integers $n, r \geq 1, k \geq 1$, by

$$c_k(n, r) = \sum_{(x, r^k)_k = 1} e(nx, r^k), \tag{1.1}$$

where $e(a, r^k) = \exp (2\pi ia/r^k)$, and the summation ranges over a k-reduced residue system (mod r). The two previous articles concerned with this sum, [3], [4], will be referred to as I and II, respectively.

In the special case $k = 1$, $c_k(n, r)$ reduces to the ordinary Ramanujan sum, $c_1(n, r) = c(n, r)$. A fundamental property of $c(n, r)$ is given by the relation,

$$c(n, r) = \Phi(n, r) \equiv \frac{\phi(r)\mu(d)}{\phi(d)} \qquad \left(d = \frac{r}{(n, r)}\right), \tag{1.2}$$

where $\phi(r)$ denotes the Euler ϕ-function and $\mu(r)$ is the Möbius function. This formula was proved by O. Hölder in 1936 [8]. Subsequent proofs have been given by Anderson and Apostol [1], Gagliardo [7], and Nicol and Vandiver [9]. The notation $\Phi(n, r)$ is due to Nicol and Vandiver, who pointed out that the function $\Phi(n, r)$ had appeared much earlier in the work of von Sterneck.

In this paper we generalize (1.2) to a relation connecting $c_k(n, r)$ and the extended totient function $\phi_k(r)$, defined as in [5] to be the number of integers in a k-reduced residue system (mod r). In particular, we prove in Theorem 1 (§2) the relation

$$c_k(n, r) = \Phi_k(n, r) \equiv \frac{\phi_k(r)\mu(d)}{\phi_k(d)} \qquad \left(d^k = \frac{r^k}{(n, r^k)_k}\right). \tag{1.3}$$

Since $\Phi_1(n, r) = \Phi(n, r)$, this relation reduces to (1.2) in the case $k = 1$. In place of generalizing the proofs of (1.2) to the case $k > 1$, we reduce the proof in the general case to that of $k = 1$.

We shall also be concerned in this paper with a basic orthogonality property of $c_k(n, r)$ proved in II in the two equivalent forms contained in (3.1) and (3.2) below. Three other forms of this relation are obtained in §3 (Theorems 6, 9, and 10). As special cases of these results, we deduce a number of arithmetical relations involving $c_k(n, r)$ and other functions.

In Theorem 12 (§4) an explicit formula is deduced for the number of solutions $N_t(n, r, k)$ of the linear congruence,

$$n \equiv x_1 + \cdots + x_t \pmod{r^k}, \tag{1.4}$$

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