SINGULAR PERTURBATIONS OF NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS RELATED TO CONDITIONAL STABILITY

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1. **Introduction.** In [3] the relationship, as $\epsilon \to 0+$, of the solutions of the real systems of differential equations

$$(1.1) \qquad \frac{dx}{dt} = f(t, x, y, \epsilon), \qquad \epsilon \frac{dy}{dt} = g(t, x, y, \epsilon)$$

$$(1.2) \qquad \frac{dx}{dt} = f(t, x, y, 0), \qquad 0 = g(t, x, y, 0)$$

where x and y are vectors of m and n components respectively, were investigated under the hypothesis that the real part of each of the characteristic roots of a certain critical matrix, the matrix $g_y(t)$ below, was negative. In a more recent work, [2] Flatto and Levinson consider (1.1) and (1.2) in the case that some of the characteristic roots of $g_y(t)$ have positive real parts; however, they consider only the case that f and g are periodic in t of period T and that (1.2) possesses a solution of period T. It is mentioned in [2] that with the aid of a suitable change of variables and a strong inequality, (1.3) and the lemma below, which are given there, that the proofs of [3] could be considerably simplified. Here we shall consider (1.1) and (1.2) under the same hypothesis as in [2] but without the assumptions of periodicity. The results obtained exhibit the boundary layer effect that is associated with systems of this type, but which do not appear in the periodic case.

The theorems given below indicate the strong similarities and the essential differences between the singular perturbation problem discussed here and the classical conditional stability problem; with respect to the latter see [1] and [4]. In particular, we find that there are "stable" initial manifolds for (1.1) which are analogous to the stable initial manifolds encountered in the problem of conditional stability. However, the behavior of those solutions which do not start on the stable initial manifolds is different for the two problems.

Let f_x denote the matrix which has $\partial f_i\ (t, x, y, \epsilon)/\partial x_j$ as the element in the i-th row and j-th column. The matrices f_y, g_x, and g_y are similarly defined.

The following hypotheses are assumed:

H1: *$x = p(t)$, $y = q(t)$ is a solution of (1.2) for $0 \le t \le T$.*

H2: *f, g, f_x, f_y, g_x, g_y are continuous if $| x - p(t)| + | y - q(t)| + \epsilon$ is sufficiently small for $0 \le t \le T$.*

Let $f_x(t) = f_x(t, p(t), q(t), 0)$ with similar definitions for $f_y(t)$, $g_x(t)$, $g_y(t)$.

H3: *There exists a non-singular real matrix $P(t) \in C'$ $(0 \le t \le T)$ such that*

$$P^{-1}(t)g_y(t)P(t) = \begin{pmatrix} B(t) & 0 \\ 0 & C(t) \end{pmatrix} \qquad (0 \le t \le T)$$

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