THE ESSENTIAL SPECTRA BELONGING TO BOUNDED AND HALF-BOUNDED POTENTIALS

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1. Let $f = f(t)$ be a continuous, real-valued function on $0 \leq t < \infty$ for which the differential equation

$$(1) \qquad x'' + (\lambda + f(t))x = 0$$

is of the limit-point type; [7; 238]. Then (1) and a boundary condition

$$(2) \qquad x(0) \cos \alpha + x'(0) \sin \alpha = 0, \qquad 0 \leq \alpha < \pi,$$

determine a self-adjoint boundary value problem on $0 \leq t < \infty$ with a spectrum $S = S_\alpha$. It is known that the essential spectrum, S', consisting of the set of cluster points of S_α is independent of α; [7; 251].

As in [4], let $\sum(\lambda_k, \lambda^k)$ denote the canonical decomposition into disjoint open intervals of the complement of the closed set S'; thus the intervals (λ_k, λ^k) are the "gaps" of the set S'. Various estimates for the size of these gaps in case f is bounded (in which case (1) is surely of the limit point type) were obtained in [4]. Moreover, sharper estimates were obtained if, in addition, f possesses a bounded first derivative or bounded first and second derivatives on $0 \leq t < \infty$. Estimates in case f possesses bounded higher derivatives can be obtained from the formulae of (**) in [2; 495].

These estimates for $\lambda^k - \lambda_k$, in terms of λ_k for large λ_k, involve a main term and a smaller correction term. It will be seen below that the latter can be omitted, so that if the main term is 0, it is possible to conclude that S' contains a half-line. For example, it will follow from (I_0) below that if $f(t)$ is bounded on $0 \leq t < \infty$ and satisfies the condition

$$(3) \qquad \liminf_{T \to \infty} T^{-1} \int_0^T \max_{|h| \leq \delta} | f(t + h) - f(t) | \, dt = 0 \quad \text{for some} \quad \delta > 0,$$

then S' contains some half-line $C \leq \lambda < \infty$. (Under the corresponding condition, one can only conclude from [4] that if there are gaps (λ_k, λ^k) in S' with large λ_k, then $\lambda^k - \lambda_k = O(1/\lambda_k)$ as $\lambda_k \to \infty$.)

The refinements of the results of [4], [2] will depend on improvements of estimates for $N = N(T, \lambda)$, the number of zeros of a solution $x = x(t, \lambda) \not\equiv 0$ of (1) on $0 < t \leq T$. These new formulae for $N(T, \lambda)$ can be used, for example, to obtain complete asymptotic expansions, in terms of $1/\lambda$, for the n-th eigenvalue of Sturm-Liouville problems associated with (1) on a finite interval $0 \leq t \leq T(< \infty)$. The proofs below will be considerably simpler that those of [4].

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