MOEBIUS INVERSION OF FOURIER TRANSFORMS

By Richard R. Goldberg and Richard S. Varga

Introduction. The classical inversion of

$$F(t) = \int_0^\infty \phi(u) \cos tu \, du \tag{*}$$

is

$$\phi(t) = \frac{2}{\pi} \int_0^\infty F(u) \cos tu \, du.$$

In this paper we present a method of inverting (*) which uses no integration whatsoever. The method consists of an application of the Moebius inversion formula combined with a variation of the classical Poisson formula from Fourier analysis. The main result is contained in Theorem 3. (*Added in proof.* It has been called to our attention that a similar result was announced by R. J. Duffin in the Bulletin of the American Mathematical Society, vol. 47(1941), p. 383.)

Theorem 3. *If* 1. $\phi(u)$ *of bounded variation on* $(0 \leq u \leq R)$ *for every* $R > 0$,

2. $\int_1^\infty | \phi(u) | \log u \, du < \infty$, *and*

3. $F(t) = \int_0^\infty \phi(u) \cos tu \, du$,

then A. $G(t) = \frac{1}{t}\left[\frac{F(0)}{2} + \sum_{k=1}^\infty (-1)^k F\left(\frac{k\pi}{t}\right)\right]$

is finite almost everywhere $(0 < t < \infty)$ *and*

B. $\phi(t) = \sum_{n=1}^\infty \mu_{2n-1} G[(2n - 1)t]$

almost everywhere $(0 < t < \infty)$.

Here the $\{\mu_n\}$ are the Moebius numbers, defined in Example 1 of Section II.

I. Two lemmas on sums.

Lemma 1. *If* 1. $\int_R^\infty | \phi(t) | \, dt < \infty$ *for every* $R > 0$,

then $\sum_{k=1}^\infty | \phi(kt) | < \infty$ *almost everywhere* $(0 < t < \infty)$.

Received March 14, 1956.