A NOTE ON HARMONIC FUNCTIONS DEFINED IN A HALF PLANE

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In this note we propose to give new proofs for two well-known theorems of potential theory, and to show that these theorems, apparently more or less unrelated, are in fact quite similar. The results in question are the sharp form of the Phragmén-Lindelöf theorem, due to Heins, and the classical Julia-Wolff theorem. These theorems, as they appear for harmonic functions, are stated as (A) and (B) below; Theorem 1 contains the connection between (A) and (B) which we have already mentioned.

As part of the proof of Theorem 1 we obtain a generalization of a well-known representation theorem for positive harmonic functions. Because of its independent interest, this result is stated as Theorem 2.

(A) Phragmén-Lindelöf-Heins Theorem. *Let $u(z)$, $z = (x, y)$, be harmonic in the half plane $y > 0$, let $\liminf_{z\to(x,0)} u \geq 0$, and suppose that*

$$\limsup_{r\to\infty} \frac{m(r)}{r} > -\infty, \qquad m(r) = \min_{|z|=r} u(z).$$

Then the limit

$$\alpha = \lim_{r\to\infty} \frac{m(r)}{r}$$

exists and $\alpha \leq 0$. Furthermore $u \geq \alpha y$, where the equality holds if and only if $u \equiv \alpha y$.

In [1] a more general result is proved, namely that statement (A) holds also for superharmonic functions (see [2]); our method of proof and the connection with the Julia-Wolff theorem seem, however, to be restricted to the harmonic version.

(B) Julia-Wolff Theorem. *Let $u(z)$ be harmonic and positive in the half-plane $y > 0$. Then the limit*

$$\beta = \lim_{z\to\infty} \frac{u}{y}$$

exists when $z \to \infty$ in any sector S: $y \geq k \mid x \mid$, $k > 0$. Moreover,

$$\lim_{z\to\infty} \frac{\partial u}{\partial x} = 0, \qquad \lim_{z\to\infty} \frac{\partial u}{\partial y} = \beta$$

as $z \to \infty$ in S. Finally, we have $u \geq \beta y$, where the equality holds if and only if $u \equiv \beta y$.

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