INVARIANT MEASURES FOR MANY-ONE TRANSFORMATIONS

By Ottis W. Rechard

1. **Introduction.** The classical individual ergodic theorem [9] is concerned with an abstract set X, a finite measure m defined on a countably additive class $\mathfrak{X}$ of measurable subsets of X with $m(X) = 1$, a real-valued integrable function f on X, and a one-one measurable and measure preserving a transformation τ of X into itself. The theorem asserts the convergence almost everywhere of the means $1/n \sum_{i=0}^{n-1} f(\tau^i x)$.

Since values of m, other than zero, do not enter into the conclusion of the theorem, it is natural to ask if for a given measurable transformation τ, there exists a measure m^* defined for sets in $\mathfrak{X}$, invariant with respect to τ, and appropriately related to m.

This question has been investigated extensively by E. Hopf [10], P. Halmos [8] and others. One of the more recent results due to M. Cotlar and R. A. Ricabarra [2] states that there exists a finite invariant measure m^* equivalent to m if and only if the set functions $m[\tau^n(A)]$, $n = 0, \pm 1, \pm 2, \cdots$ are uniformly absolutely continuous with respect to m. (In their paper, Cotlar and Ricabarra state and prove the result for more general groups of transformations than the cyclic group $\{\tau^n\}$.)

In 1940, J. L. Doob [3] and later during the war, but independently, F. Riesz [11] showed that the conclusions of the ergodic theorem remain true if the requirement that τ be one-one is dropped, and it is only assumed that the inverse image of every measurable set is a measurable set of the same measure. It thus becomes of interest to know if for a given many-one transformation τ, there exists a measure m^* defined for sets in $\mathfrak{X}$, invariant with respect to τ (in the above sense) and suitably related to m. The purpose of this paper is to investigate this question. The main result is contained in Theorem 2, which in some respects can be regarded as a generalization to the many-one case of the theorem of Cotlar and Ricabarra. However, the method of proof used here is quite different from that employed by these two authors. In §4, Theorem 2 is applied to the proof of a theorem due to Dunford and Miller, §5 deals briefly with the problem of extending Theorem 2 to more general semi-groups of transformations, and §6 contains examples of invariant measures for some particular transformations of the unit interval.

2. **Definitions and preliminary results.** Throughout this paper, τ will represent a transformation (not necessarily one-one) of X into itself. If the complete inverse image of every measurable set is itself measurable, then τ will be called

Received January 9, 1956. Presented to the American Mathematical Society September 2, 1954. The author is indebted to S. Ulam for many helpful and stimulating conversations on the subject of this paper.