EXTREME PROPERTIES OF EIGENVALUES OF A HERMITIAN TRANSFORMATION AND SINGULAR VALUES OF THE SUM AND PRODUCT OF LINEAR TRANSFORMATIONS

By Ali R. Amir-Moéz

Introduction. The purpose of this paper is to study the relations between the singular values of a sum or product of linear transformations A, B and the singular values of A and of B. A closely related problem is that of the relations between the eigenvalues of AB or $A + B$ and the eigenvalues of A and of B, when A and B are Hermitian. In the latter problem, there are two known results. These are due to Weyl in 1912 and Lidski in 1950. We shall derive a more general set of inequalities which includes these results as special cases.

Section 1 is devoted to definitions and notations. In §2, we give a general extreme characterization of certain functions of the eigenvalues of Hermitian transformation. This is a generalization of a recent theorem of Wielandt [11]. This theorem is used in §3 to derive the new inequalities. Section 4 contains a discussion and comparison of the various inequalities.

1. **Definitions and notations.** We denote by E_n the unitary space of dimension n. The inner product of two vectors will be indicated by (x, y). If $x_1, \cdots, x_k$ are vectors, then $[x_1, \cdots, x_k]$ denotes the subspace spanned by them. In certain formulas, we use the abbreviation o.n. for orthonormal. If M and N are subspaces of any vector space, then $M \oplus N$ denotes the subspace spanned by M and N. In a unitary space, $M^\perp$ denotes the space of vectors which are orthogonal to M. If $M \subset N$, then $N \ominus M$ denotes $N \cap M^\perp$.

The symbol $[a_{ij}]$ denotes the matrix with elements a_{ij}. By det $[a_{ij}]$ we mean the determinant of the matrix $[a_{ij}]$.

The Gram determinant of a set $\{x_1, \cdots, x_k\}$ of vectors is det $[(x_i, y_j)]$. The set $\{x_p\}$ is linearly dependent if and only if its Gram determinant is zero.

By a linear transformation on a vector space S, we mean a linear transformation with domain S and values in S. If A is a linear transformation on E_n, its adjoint is denoted by A^* and we have $(Ax, y) = (x, A^*y)$. A Hermitian transformation A is one for which $A = A^*$. A Hermitian transformation A has real eigenvalues; if in addition the eigenvalues are non-negative, the transformation A has a unique square root $\sqrt{A}$ with the properties

$$(1) \qquad (\sqrt{A})^2 = A, \qquad\qquad (2) \qquad \sqrt{A} \text{ is non-negative.}$$

For any linear transformation A on E_n, A^*A is clearly non-negative. The non-negative square roots of the eigenvalues of A^*A are called the singular values of A.

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