HOMOGENEITY PROBLEMS IN THE THEORY OF ČECH COMPACTIFICATIONS

By Walter Rudin

Introduction. If X is a completely regular topological space, there exists a space βX, the so-called Čech compactification of X, which is characterized by the following three properties: βX is a compact (bicompact, in the older terminology) Hausdorff space, X is a dense subset of βX, and every bounded continuous real-valued function on X can be extended to a continuous function on βX [1; 831].

A topological space X is homogeneous if to every pair of points p and q of X there exists at least one homeomorphism of X which carries p to q. In the seminar on function spaces which was part of the Institute on Set Theoretic Topology held at Madison during the summer of 1955, the question was raised whether the homogeneity of X implies the homogeneity of the space $X^* = \beta X - X$. In the present paper it is shown that the answer to this question is negative (under the assumption of the continuum hypothesis) even if X is a countable discrete space (Theorem 4.4). This in turn implies that X^* fails to be homogeneous whenever X is locally compact but not sequentially compact (Theorem 4.5).

We choose the set N of all positive integers as our countable discrete space, and construct βN as the set of all ultrafilters on N. This purely set-theoretic construction of βN leads to a fairly detailed description of that space, and in particular of the space $N^* = \beta N - N$. The latter is the main object of study and is found to be a compact Hausdorff space with 2^c points in which there are exactly c open-closed subsets (the letter c stands for the cardinal number of the continuum); these sets form a basis for the topology of N^*, and any two non-empty open-closed subsets are homeomorphic; the intersection of any countable family of open sets is either empty or contains a non-empty open set; and N^* is almost homogeneous, in the sense that if any point and any open set of N^* are given, there is a homeomorphism of N^* which carries the given point into the given open set. These properties do not depend on the continuum hypothesis.

If the continuum hypothesis is true, then N^* contains P-points, using the terminology of Gillman and Henriksen [2; 343] (see Definition 4.1), and it is this fact which leads almost immediately to the conclusion that N^* is not homogeneous.

The space βN has precisely c homeomorphisms; these are in natural one-to-one correspondence with the permutations of N. On the other hand, the con-

Received January 16, 1956