# $n$-VALUED IRREGULAR BOREL MEASURES

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1. **Introduction.** In theorems concerning a Borel measure in published mathematical research, we so often find the condition "regular" imposed on the measure. A reason for this is the fact that irregular measures on general Borel sets cannot be computed by knowing the values of the measures on the compact or on the open Borel sets of the space or both [3; 224]. We ask when and on what spaces it might be possible to rewrite some of these theorems without the word "regular" in the hypotheses. We are thus led to investigate two questions: what are necessary and sufficient conditions on topological spaces for the existence of irregular Borel measures? And, if we cannot compute the values of irregular measures by means of the measures on the compact nor on the open Borel sets, is there a class of Borel sets other than the compact and open sets by means of which these computations are possible? In this paper we answer the above two questions for $n$-valued, totally finite, irregular Borel measures.

There are good reasons for the study of $n$-valued measures. First, while some work has been done concerning general irregular Borel measures (see [4]), they are still difficult to handle. The $n$-valued measure is somewhat simpler to study than a general measure. Second, progress in the study of the $n$-valued measures may reveal clues for the ground work of the study of general measures. And, third, counter examples using $n$-valued measures are easily constructed which show that certain natural conjectures in the theory are false.

Let $X$ be a topological space, let the Borel sets of $X$ be the $\sigma$-algebra $\mathcal{B}$ generated by the class $\mathcal{C}$ of all compact sets of $X$, and let $\mathcal{D}$, $\mathcal{U}$, and $\mathcal{V}$ denote the classes of all closed, open, and open Borel sets of $X$, respectively.

Any measure whose domain of definition includes the Borel sets of $X$ is said to be a Borel measure on $X$. If $\mu$ is a Borel measure on $X$, then $\mu$ is outer [inner] regular at a Borel set $A$ if and only if

$$\mu(A) = \inf\{\mu(V): A \subset V \in \mathcal{V}\}\ [= \sup\{\mu(C): A \supset C \in \mathcal{C}\}];$$

otherwise $\mu$ is outer [inner] irregular at $A$. If $\mu$ is both outer and inner regular at a Borel set $A$, $\mu$ is said to be regular at $A$; otherwise $\mu$ is irregular at $A$. We shall say that a measure $\mu$ is $n$-valued on a space $X$ if and only if $\mu$ assumes exactly $n$ distinct values on the Borel sets of $X$, where $n$ is a fixed positive integer.

In this paper we first concentrate our study to properties of the $n$-valued irregular measures, themselves, and then later, in §3, to the spaces which admit them.

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