IDEALS IN MULTIPLICATIVE SEMI-GROUPS OF CONTINUOUS FUNCTIONS

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1. **Introduction.** Let C be an abstract multiplicative semi-group which is known to be the semi-group of the continuous functions vanishing at infinity on some locally compact space X. The case in which X is compact was studied by Milgram [3] who showed a $1 - 1$ correspondence between the closed sets F in X and a class of ideals I_F in C defined purely algebraically. In terms of X, the ideal I_F turned out to be the set of functions vanishing in the neighborhood of F. In the case of compact X, and for a certain class of sub semi-groups, B, of C, now considered as an abstract topological semi-group, a $1 - 1$ correspondence was again obtained by one of us [6]. The correspondence associated to a closed set F an ideal J_F, defined in terms of B and not X. In terms of X, J_F turned out to be the set of functions vanishing on F. In this paper we characterize those semi-groups B of C in which correspondences of the above type exist. Our results can be applied to regular Banach algebras. One application is the following. Let B_i be a semi-simple regular Banach algebra with space of regular maximal ideals $\mathfrak{M}_i$, $i = 1, 2$. If B_1 is algebraically isomorphic to B_2 either as a multiplicative semi-group or as a semi-group under the operation $f \circ g = f + g - fg$, then $\mathfrak{M}_1$ is homeomorphic to $\mathfrak{M}_2$. For a discussion of similar theorems involving the continuous functions on X given abstractly in terms of other operations see Kaplansky [2].

2. **Notation.** Let $C = C(X)$ be the Banach algebra of all real (or all complex) continuous functions which vanish at infinity on a locally compact Hausdorff space X. In §§3, 4 we consider C as an abstract multiplicative semi-group, and in §5 as an abstract topological multiplicative semi-group with topology consistent with that given by the Banach space norm. All concepts defined below in terms of X are shown in the appendix (§6) to be entirely definable in terms of C as it is considered in the appropriate section.

We are concerned with various types of semi-groups B in C. Let F_1, F_2 be any two disjoint closed sets in X with F_1 compact and t_1 any point not in F_2. B is called *regular* if there always exists $f \in B$ such that $f(t) = 0, t \in F_2$, $f(t_1) \neq 0$. B is called *normal* if for every $\epsilon > 0$ there exists $f \in B$ such that $f(t) = 0, t \in F_2$ and $| f(t) - 1 | < \epsilon, t \in F_1$. B is called a *Šilvov semi-group* if there exists $f \in B$ such that $f(t) = 1, t \in F_1$, $f(t) = 0, t \in F_2$. B is called a *general Urysohn semi-group* if B^c is a Šilov semi-group. (For a set A in either X or C, its closure is denoted by A^c).

A subset I of the semi-group B in C is called an *ideal* if $f \in I$, $g \in B$ implies that

Received July 1, 1955.