DIRECTED LIMITS ON RINGS OF CONTINUOUS FUNCTIONS

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1. **Introduction.** Let X denote an arbitrary set. Let βX^D denote the set of all ultrafilters (proper maximal dual ideals) of subsets of X. Presently, under certain conditions, the real-valued function f defined on X will be assigned a real-valued directed limit $\bar{f}(\alpha)$ at the point α of βX^D.

Let (X, T) denote the set X with a completely regular topology T. Let $\upsilon(X, T)$ and $\beta(X, T)$ denote the Hewitt Q-space and Stone-Čech extensions of (X, T). For (X, T), and similarly for other spaces, let $C(X, T)$ and $C^*(X, T)$ denote in turn the spaces of all real-valued and of all bounded, real-valued functions, defined and continuous on (X, T). Our first purpose is to indicate, in brief detail, the use of βX^D and of its directed limits in representing the spaces (X, T), along with their accompanying topological and function spaces, for all completely regular topologies T. Secondly, to illustrate the utility of this representation, we will prove several new facts concerning the maximal ideals of $C(X, T)$. For example, it will be shown that the maximal ideals of $C(X, T)$ are exactly the subsets $M(\alpha) = [f \in C(X, T) \mid \overline{f \cdot g}(\alpha) = 0, \text{all } g \in C(X, T)]$ determined by elements α of βX^D.

2. **Spaces βX^D and $\beta(X, T)$.** Let (X, T) be as above. Let $\mathfrak{A}$ denote the set of all finite normal coverings [5; 6] of (X, T) by open sets. A dual ideal in the lattice of open sets of (X, T) is said to be under $\mathfrak{A}$ if it contains at least one open set from each of the coverings that constitute $\mathfrak{A}$. Any ideal which is under $\mathfrak{A}$ contains an ideal which is minimal with respect to this property [5; 288]. Let $\beta(X, T)$ with elements $\alpha^T, \beta^T, \cdots$, denote the set of all such ideals minimal with respect to being under $\mathfrak{A}$. In case T is the discrete topology, the set $\beta(X, T)$ with elements α^T is identical with the set βX^D of all ultrafilters α of subsets of X. In any case, to each α^T in $\beta(X, T)$, for each subset U of (X, T) which it contains, assign as a neighborhood of α^T in $\beta(X, T)$, the set of all elements of $\beta(X, T)$ which contain U. The space $\beta(X, T)$, thus topologized, is (homeomorphic to) the Stone-Čech compactification of (X, T) (see [5, Ex. 10.3], [1], [7]).

To these familiar concepts, we add the following: the real-valued function f defined on the set X will be said to have a directed limit of (necessarily unique) value $\bar{f}(\alpha^T)$ at the point α^T of $\beta(X, T)$, provided a real number $\bar{f}(\alpha^T)$ exists such that, for arbitrary positive number ϵ, a subset U_ϵ of (X, T) is found in α^T on which $f(x)$ differs from $\bar{f}(\alpha^T)$ by less than ϵ. The following statement is then quite obvious.

Theorem 2.1. *Each element f of $C^*(X, T)$ has a directed limit $\bar{f}(\alpha^T)$ at each point α^T of $\beta(X, T)$ and the functions $\bar{f}$ thus defined constitute $C[\beta(X, T)]$.*

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