REDUCTION OF EQUATIONS TO NORMAL FORM IN FIELDS OF CHARACTERISTIC p

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1. **Introduction.** In [2] the equation of degree 5 with coefficients in a field F of characteristic $p = 2, 3, 5$ was reduced to *principal* and *normal* form by means of Tschirnhaus transformations. Dickson [1, Chapter XII] gave similar results for the quintic in case F is of characteristic 0 which also hold for F of characteristic $p \geq 7$. The present paper extends those results of [2] to equations of degree n, $n \geq 6$, with coefficients in a field F of characteristic $p \geq 2$. Theorem 1 treats the *principal* form while Theorem 2 yields the *normal* form, both for $p \geq 7$. Theorems 3 and 4 relate to the distinctly different situations for $0 < p < 7$. Throughout, certain differences are evident depending on whether $p \nmid n$ or $p \mid n$. The transformation of lowest degree which will accomplish the desired reduction is preferred and usually employed.

2. **Preliminary.** Let $x_1, x_2, \cdots, x_n$ be the roots of the equation

$$F(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0 \qquad (a_i \in F), \tag{2.1}$$

where F is a field of characteristic p. A Tschirnhaus transformation [3] of (2.1) into

$$F(y) = y^n + b_1 y^{n-1} + \cdots + b_n = 0 \tag{2.2}$$

is taken in the form

$$y = cx^{n-1} + dx^{n-2} + \cdots + f. \tag{2.3}$$

The coefficients of (2.2) are known by symmetric functions in terms of the coefficients of (2.3) which are taken as parameters.

Newton's identities are given for later use by

$$S_k = -a_1 S_{k-1} - a_2 S_{k-2} - \cdots - a_{k-1} S_1 - k a_k, \tag{2.4}$$

where S_k denotes the sums of the k-th power of the roots of (2.1) in terms of the a_i. In particular

$$\begin{aligned} &S_1 = -a_1, \quad S_2 = a_1^2 - 2a_2, \quad S_3 = -a_1^3 + 3a_1a_2 - 3a_3, \\ &S_4 = a_1^4 - 4a_1^2a_2 + 2a_2^2 + 4a_1a_3 - 4a_4, \\ &S_5 = -a_1^5 + 5a_1^3a_2 - 5a_1^2a_3 + 5a_1a_4 - 5a_1a_2^2 + 5a_2a_3 - 5a_5, \\ &S_6 = a_1^6 - 6a_1^4a_2 + 6a_1^3a_3 - 6a_1^2a_4 + 9a_1^2a_2^2 \\ &\qquad - 12a_1a_2a_3 + 6a_1a_5 - 2a_2^3 + 6a_2a_4 + 3a_3^2 - 6a_6, \end{aligned} \tag{2.5}$$

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