CLASS NUMBER FORMULAS FOR QUADRATIC FORMS OVER $GF[q, x]$

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1. **Introduction.** Let $q = p^z$, p an odd prime, and let $GF[q, x]$ denote the set of polynomials in the indeterminate x with coefficient in the finite field $GF(q)$. In a recent paper [2], Byers has proved several class-number relations for "definite" binary quadratic forms with coefficients in $GF[q, x]$. The formulas are obtained by setting up a correspondence between classes of quadratic forms and classes of bilinear forms; the method is analogous to that used by Kronecker [6]. Let $h(\Delta)$ denote the number of classes of quadratic forms of discriminant Δ, where $\Delta \in GF[q, x]$. Then in particular for $\deg \Delta = 2m + 1$, Byers has proved the formula [1, Theorem 13]

$$\sum_{\deg R \leq m} h(R^2 - \Delta) = \sum_{\substack{J \mid \Delta \\ \deg J > m}} | J | \qquad (| J | = q^{\deg J}), \tag{1.1}$$

where the summation on the left is over all $R \in GF[q, x]$ of degree $\leq m$, including $R = 0$, while the summation on the right is restricted to *primary* divisors of Δ that are of degree $> m$. The case $\deg \Delta$ even is somewhat more complicated and is covered by two additional formulas [2, Theorems 14, 15] which employ a modified class number $h'(\Delta)$, which reduces to $h(\Delta)$ when Δ is of odd degree.

In the present paper we prove these class number formulas by a different method suggested by the "singular series" for sums of squares in $GF[q, x]$. Indeed by this method we are able to evaluate such sums as

$$\sum_{U_1, \cdots, U_s} h'(\alpha_1 U_1^2 + \cdots + \alpha_s U_s^2 - \Delta), \tag{1.2}$$

where $\alpha_1, \cdots, \alpha_s$ are fixed non-zero numbers of $GF(q)$, the summation is over all $U_i \in GF[q, x]$ of degree $< m$ or $\leq m$, and Δ is of degree $2m + 1$ or $2m$. For $s = 1$, (1.2) reduces to the left member of (1.1). In general the results (Theorem 5 below) are simpler for odd s; in this case the sum is expressed in terms of divisor functions, while for even s the sum is expressed in terms of the Artin numbers defined in (4.8) below. The sum (1.2) with $h'(\Delta)$ replaced by $h(\Delta)$ is also evaluated, but the result (Theorem 6) is more complicated.

In the next place (§6) we show how to evaluate the sum

$$\sum_{U} h(U)h(\Delta - U), \tag{1.3}$$

where, say, $\deg \Delta = 2m + 1$ and the summation is over a certain set of q^{2m+1} polynomials of degree $2m + 1$. The sum is evaluated in terms of divisor functions. Incidentally, the method used to evaluate (1.2) and (1.3) also enables

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