THE SHIFT OPERATOR FOR NON-STATIONARY STOCHASTIC PROCESSES

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1. **Introduction.** It is well known that the theory of stationary (wide sense) random functions is closely connected with the theory of groups of unitary operators on Hilbert space. This connection is based on the fact that the "shift" operators $V_s: x(t) \rightarrow x(t + s)$ are unitary if considered on the appropriate Hilbert space. The purpose of this paper is to investigate these shift operators in the non-stationary case and especially to determine what types of random functions result when these operators are self-adjoint or normal.

Clearly the shift operators can not be defined in all cases. In §2 we investigate the conditions under which these operators are defined. §3 also contains preliminary material. The main result is a representation theorem (Theorem 3A) which is essentially an operator-theoretic reformulation of a theorem of Karhunen [5]. The proof, which is only sketched, is somewhat shortened by making use of operator theory.

The main section of the paper, §4, is devoted to the case in which the shift operators are normal. We consider only the case $t \geq 0$. It turns out that to require each V_s to be normal is too restrictive. For example, in the study of stationary processes $x(t)$ for $t \geq 0$ the first thing that one does is extend the definition of $x(t)$ to negative t values by putting $x(-t) = V_t^* x(0)$, that is, the adjoints of the shift operators are introduced. The Hilbert space in question is then generated by $\{x(t); -\infty < t < \infty\}$ or equivalently by $\{V_t V_s^* x(0); t, s \geq 0\}$ and not by $\{x(t); t \geq 0\}$ alone. These considerations lead us to the definition of §4. Theorem 5A shows that $e^{\lambda t}$ $(\lambda = \lambda_1 + i\lambda_2)$ plays the same role in the present theory as $e^{i\lambda_2 t}$ plays in the stationary case. Theorem 5B states that the Hilbert space in question is precisely that generated by $\{N_t N_s^* x(0); t, s \geq 0\}$ where the N_t's are the "extended" shift operators. This is completely analogous to the stationary case. We then characterize the covariance function of such random functions and finally prove a mean ergodic theorem for this class of processes.

In the last section the results of §4 are specialized to the self-adjoint case and yield Loève's theory of random functions of convex exponential order.

The following notations will be used in this paper. $(\Omega, \mathfrak{F}, P)$ will denote a probability space and $L^2(\Omega)$ the complex Hilbert space of complex valued $\mathfrak{F}$-measurable square integrable functions on $(\Omega, \mathfrak{F}, P)$. If T is a subset of the real numbers, then a mapping $x : T \rightarrow L^2(\Omega)$ will be called a random function and will be written $\{x(t); t \in T\}$. Let $L(x)$ be the smallest linear manifold of $L^2(\Omega)$

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