# THE ESSENTIAL SPECTRUM AND AVERAGES OF THE POTENTIAL

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1. Let $q(t)$ be real-valued and continuous on $0 \leq t < \infty$. When

$$x'' + (\lambda + q(t))x = 0 \tag{1}$$

is of the limit-point type in the sense of Weyl [15; 238], the set of cluster points $S'$ of the spectrum $S(\alpha)$ of the self-adjoint operator belonging to (1) and to a boundary condition $x(0) \cos \alpha - x'(0) \sin \alpha = 0$ is called the essential spectrum of (1). The $\lambda$-set $S'$ is independent of $\alpha$; [15; 251–252].

It is known that if $q(t)$ is "small" at $t = \infty$, then

$$S' \text{ is the half-line } [0, \infty). \tag{2}$$

For example, (2) holds if $q$ satisfies any one of the following conditions:

$$q(t) \to 0 \quad \text{as} \quad t \to \infty, \tag{$3_1$}$$

$$\int^{\infty} | q(t) |^p \, dt < \infty \quad \text{for some} \quad p \geq 1, \tag{$3_2$}$$

or

$$\int^{\infty} q(t) \, dt = \lim_{T \to \infty} \int^{T} q(t) \, dt \text{ converges (conditionally).} \tag{$3_3$}$$

(A proof for the sufficiency of $(3_1)$ for (2) was given in [3] (a simpler proof follows from [4]); the sufficiency of $(3_2)$, when $p = 2$, is given in [12] and can be concluded, in general, from [10], [1] and [4]; for the sufficiency of $(3_3)$, see [14]).

The object of this paper is to discuss the relationship of the essential spectrum $S'$ of (1) to the behavior of certain averages of $q(t)$. In the first part (Sections 1–4), the results will involve the behavior, as $t \to \infty$, of the function

$$Q(t) = M - \int_0^t q(s) \, ds, \tag{4}$$

where $M$ is a constant. In this direction, one has the following theorem:

(i) *To the continuous function $q(t)$, let there belong a number $M$ with the property that* (4) *satisfies*

$$\liminf_{T \to \infty} T^{-1} \int_0^T Q^2(t) \, dt = 0. \tag{5}$$

*Then* (1) *is of the limit-point type and*

$$S' \supset [0, \infty). \tag{6}$$

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