FINITE ABELIAN GROUPS WITH ISOMORPHIC GROUP ALGEBRAS

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Introduction. If $\mathcal{G}$ is a group of order g and $\mathfrak{F}$ is a field then it is possible to form, in a well-known fashion, an algebra $\mathfrak{A}(\mathcal{G})$ of order g over $\mathfrak{F}$ called the group algebra (or group ring) of $\mathcal{G}$ over $\mathfrak{F}$. At the Michigan Algebra Conference in the summer of 1947, R. M. Thrall proposed the following problem: Given the group $\mathcal{G}$ and the field $\mathfrak{F}$, determine all groups $\mathcal{K}$ such that $\mathfrak{A}(\mathcal{K})$ is isomorphic with $\mathfrak{A}(\mathcal{G})$ over $\mathfrak{F}$. Perlis and Walker [2] restated the problem: Given the groups $\mathcal{G}$ and $\mathcal{K}$ of order g, find all fields $\mathfrak{F}$ such that $\mathfrak{A}(\mathcal{K})$ is isomorphic with $\mathfrak{A}(\mathcal{G})$ over $\mathfrak{F}$. They presented a complete solution of the problem for the case in which $\mathcal{G}$ is Abelian and $\mathfrak{F}$ has characteristic 0 or a prime not dividing g. In this paper we shall complete the solution of the Abelian case by solving the problem when $\mathfrak{F}$ has characteristic p which divides g.

The problem which arises when the characteristic of $\mathfrak{F}$ divides the order of $\mathcal{G}$ is complicated by the fact that $\mathfrak{A}(\mathcal{G})$ is no longer semisimple. Thus the methods of this paper differ sharply from those employed by Perlis and Walker who were working with direct sums of fields.

In Section 1 we shall exhibit some relations between subgroups of a group and certain ideals of its group ring, while in Section 2 we shall prove the key result (Theorem 2): If $\mathcal{G}$ is an Abelian p-group and $\mathfrak{F}$ is of characteristic p, then $\mathfrak{A}(\mathcal{K})$ is isomorphic with $\mathfrak{A}(\mathcal{G})$ if and only if $\mathcal{K}$ is isomorphic with $\mathcal{G}$. These results are combined in Section 3 with the results of Perlis and Walker to yield the solution to the Abelian portion of Thrall's problem.

1. **Subgroups and Ideals.** Let $\mathcal{G}$ be a group of order g, $\mathfrak{F}$ be a field (of arbitrary characteristic), and $\mathcal{H}$ be a subgroup of $\mathcal{G}$ of order h consisting of elements $H_1 = 1, H_2, \cdots, H_h$. Select q elements of $\mathcal{G}$, $Q_1, \cdots, Q_q$, so that

$$\mathcal{G} = Q_1\mathcal{H} + \cdots + Q_q\mathcal{H} = \mathcal{H}Q_1 + \cdots + \mathcal{H}Q_q$$

where $qh = g$, and form the set L of the $g - q$ elements $Q_i(H_j - 1)$, $i = 1, \cdots, q$ and $j = 2, \cdots, h$, of the group algebra $\mathfrak{A}(\mathcal{G})$.

(1) L is a set of linearly independent elements (over $\mathfrak{F}$) of $\mathfrak{A}(\mathcal{G})$ since the g elements Q_iH_j, $i = 1, \cdots, q$ and $j = 1, \cdots, h$ form a basis for $A(G)$.

(2) The elements of L form a basis for a left ideal $\mathfrak{L}$ of $\mathfrak{A}(\mathcal{G})$ since

$$G_nQ_i(H_j - 1) = Q_rH_m(H_j - 1) = Q_r(H_k - H_m) = Q_r(H_k - 1) - Q_r(H_m - 1).$$

We say that $\mathfrak{L} = \mathfrak{L}(\mathcal{H})$ is the left ideal of $\mathfrak{A}(\mathcal{G})$ associated with the subgroup $\mathcal{H}$ of $\mathcal{G}$.

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