THE FORMAL POWER SERIES FOR LOG $e^x e^y$

By Karl Goldberg

1. **Introduction.** That log $e^x e^y$ can be expressed as a sum of x, y, and the commutators ($xy - yx$, and so on) of x and y was first shown by J. E. Campbell [2] in 1898, H. F. Baker [1] in 1905, and F. Hausdorff [7] in 1906. The expansion in terms of the commutators has been used in such fields as group theory [8] and differential equations [9].

To this day the determination of the coefficients of the commutators is difficult; only scattered results are available when the degree of the commutator is greater than six.

It might be useful, therefore, to investigate log $e^x e^y$ as a formal power series in the non-commuting variables x and y, with the hope that the information gained in such an investigation will be of use in the problem of the commutator coefficients. An algorithm due to E. B. Dynkin [3] may prove useful for this purpose. See also D. Finkelstein [4] who gives an expression for log $e^x e^y$, in terms of certain symbolic operators, which may be regarded either as a commutator or a power series expansion.

The major result of this paper is a formula for the coefficients in the formal power series stated in two different ways. Theorem 1 gives these coefficients in terms of certain fixed polynomials, and any specified coefficient may be computed easily from this form. Theorem 2 gives a generating function for these coefficients from which certain coefficients are obtained as a sum of Bernoulli numbers.

2. **Statement of the Theorems.** In a formal power series in non-commutative variables x and y the general term beginning with a power of x is

$$W_x \equiv W_x(s_1, s_2, \cdots, s_m) = x^{s_1} y^{s_2} \cdots (x \vee y)^{s_m} \tag{1}$$

where $s_1 s_2 \cdots s_m \neq 0$ and $(x \vee y)^{s_m}$ denotes x^{s_m} if m is odd and y^{s_m} if m is even. W_y is similarly defined.

In the formal power series for log $e^x e^y$ we denote the coefficient of $W_x(s_1, \cdots, s_m)$ by $c_x \equiv c_x(s_1, \cdots, s_m)$, and that of W_y by c_y.

The main theorems of this paper can now be stated:

Theorem 1.

$$c_x(s_1, \cdots, s_m) = (-1)^{n-1} c_y = \int_0^1 t^{m'}(t-1)^{m''} G_{s_1}(t) \cdots G_{s_m}(t)\, dt$$

where $n = \sum_{i=1}^m s_i$, $m' = [m/2]$, $m'' = [(m-1)/2]$ and the polynomials $G_s(t)$ are defined recursively by $G_1(t) = 1$ and $sG_s(t) = d/dt\, t(t-1)G_{s-1}(t)$ for $s = 2, 3, \cdots$. Also $c_x = c_y$ if m is odd.

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