ARITHMETIC PROPERTIES OF BERNOULLI NUMBERS OF HIGHER ORDER

By F. R. Olson

1. **Introduction.** The Bernoulli numbers of order k may be defined by means of [7; Chapter 6]

$$\left(\frac{x}{e^x - 1}\right)^k = \sum_{m=0}^{\infty} \frac{x^m}{m!} B_m^{(k)} \qquad (|x| < 2\pi). \tag{1.1}$$

For $k = 1$ it is customary to write $B_m^{(1)} = B_m$. Moreover $B_{2s+1} = 0$ for $s \geq 1$. In the first part of this paper we consider some divisibility properties of $B_m^{(k)}$. S. Wachs [8] has proved a result equivalent to

$$B_{p+2}^{(p+1)} \equiv 0 \pmod{p^2}, \tag{1.2}$$

where p is a prime ≥ 3. This has been sharpened [1] to

$$B_{p+2}^{(p+1)} \equiv \frac{p^3}{6} \pmod{p^4} \qquad (p \geq 5). \tag{1.3}$$

We prove in §4 that

$$B_{p+2}^{(p+1)} \equiv \frac{p^4}{4} - \frac{p^3}{6}(p-1)! \pmod{p^5} \qquad (p \geq 5). \tag{1.4}$$

Also in [1] it was proved that

$$B_p^{(p)} \equiv -\frac{p^2}{2}(p-1)! \pmod{p^5} \qquad (p \geq 5). \tag{1.5}$$

This we extend in §3 to

$$B_p^{(p)} \equiv -\frac{p^2}{2}(p-1)! + \frac{p^5}{36} B_{p-3} \pmod{p^6} \qquad (p \geq 7). \tag{1.6}$$

Nörlund [7; Chapter 6] considered generalized Bernoulli numbers, those of positive order being defined by

$$\prod_{i=1}^{k} \frac{\omega_i x}{e^{\omega_i x} - 1} = \sum_{m=0}^{\infty} \frac{x^m}{m!} B_m^{(k)}[\omega_1, \omega_2, \cdots, \omega_k],$$

and those of negative order by

$$\prod_{i=1}^{h} \frac{e^{\omega_i x} - 1}{\omega_i x} = \sum_{m=0}^{\infty} \frac{x^m}{m!} B_m^{(-h)}[\omega_1, \omega_2, \cdots, \omega_h].$$

Received February 17, 1955. This paper constitutes a part of a doctoral thesis submitted to Duke University and prepared under the direction of Professor L. Carlitz.