GRAPHS AND MATCHING THEOREMS

By Oystein Ore

The origin of the present theory can be found in the simple theorems by Miller [17] and Chapman [4] stating that in finite groups common representatives exist for the left and right co-set expansions. This result has given rise to various generalizations, notably the graph formulations by van der Waerden [24] and König [13]. Closely related, but of a somewhat different type, is the theorem by P. Hall [9] giving the conditions for selecting a distinct set of representatives for a family of sets. Recently this theorem has been extended by R. Radó [18], M. Hall [8] and by Everett and Whaples [6] to a form involving infinite families of sets. A further bibliography can be found in a recent paper by W. Maak [14].

The purpose of the present paper is both to consider these questions from a more systematic point of view and to solve them under the general form that one determines the exact number of matchings or representatives which can be found rather than to ask primarily for the conditions under which a complete set can be found. For arbitrary families of sets the main results are contained in Theorem 13 from which the other results mentioned will follow. It is most convenient to formulate these problems in terms of bipartite graphs as first suggested by König [13], and thus we also arrive at a number of rather interesting properties of such graphs. An alternate presentation is in terms of matrices and so we obtain a normal form of matrices under permutation operations.

The principles used in this paper are applicable to general graphs and to arbitrary factors of such graphs as will be shown in a later paper. One obtains as a consequence general criteria for the existence of subgraphs and also specific factorisations of various types of graphs.

1. **Bipartite graphs.** In the following let

$$V = \{a_i\}, \qquad i \in M \tag{1}$$

and

$$V' = \{a'_j\}, \qquad j \in M' \tag{2}$$

be two arbitrary sets with elements a_i and a'_j and index sets M and M' respectively. We shall construct a *bipartite graph* G with V and V' as vertex sets by joining some of the elements a_i and a'_j in (1) and (2) by *edges* $(a_i\,,\, a'_j)$. It may be assumed that each a_i is *edge connected* to some a'_j in this manner and conversely, although this condition is not essential. More incisive is the following

Received March 7, 1955.