THE HAUSDORFF-BERNSTEIN-WIDDER THEOREM FOR SEMI-GROUPS IN LOCALLY COMPACT ABELIAN GROUPS

By A. E. Nussbaum

1. **Introduction.** In this paper we give a generalization of the Hausdorff-Bernstein-Widder theorem for semi-groups contained in a locally compact Abelian group. The methods used to prove this theorem are in essence the methods used to prove the classical moment problem theorems by Hilbert space techniques. This method has also recently been used by A. Devinatz [2] to obtain results similar to ours for Euclidean spaces.

The Hausdorff-Bernstein-Widder theorem for the real line states:

A real-valued function $f(x)$ defined on $0 \leq x < \infty$ can be represented by a Laplace-Stieltjes integral

$$f(x) = \int_0^\infty e^{-xt}\, d\alpha(t)$$

where $\alpha(t)$ is a bounded non-decreasing function of bounded variation if and only if $f(x)$ is a completely monotonic function on $[0, \infty)$.

A real-valued function $f(x)$ defined on the interval $[0, \infty)$ is said to be completely monotonic if all the differences

$$(I - T_\delta)^n f(x) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} f(x + k\delta) \geq 0,$$

for all non-negative integers n, all $x \geq 0$ and all $\delta \geq 0$, where I is the identity operator and T_δ the translation operator $T_\delta f(x) = f(x + \delta)$.

To prove our theorem we need a generalization of B. v. Sz. Nagy's and E. Hille's theorem on the spectral representation of semi-groups of self-adjoint operators acting on a Hilbert space.

We therefore begin with the spectral representation theorem for semi-groups of self-adjoint operators on a Hilbert space where the parameter semi-group involved is a *full semi-group*. By this we mean the following:

Definition 1. A semi-group $\mathfrak{S}$ is called a *full semi-group* if

(a) $\mathfrak{S}$ can be embedded in a locally compact Abelian group $\mathfrak{G}$,

(b) $\mathfrak{S}$ is measurable with respect to the Haar measure of $\mathfrak{G}$ and

(c) every non-empty open set in $\mathfrak{S}$ has non-zero measure.

The method of proof is that used by R. S. Phillips [6] in a similar connection.

2. **Semi-groups of self-adjoint operators.** Let $\mathfrak{S}$ be a full semi-group and $\{T_x\}$ a uniformly bounded self-adjoint representation of $\mathfrak{S}$. More explicitly

(a) T_x is a bounded self-adjoint operator acting on a Hilbert space $\mathfrak{H}$ for every $x \in \mathfrak{S}$

Received May 2, 1955.