ON THE NUMERICAL INTEGRATION OF A PARABOLIC DIFFERENTIAL EQUATION SUBJECT TO A MOVING BOUNDARY CONDITION

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1. **Introduction.** A number of physical problems are governed by a parabolic differential equation in conjunction with a moving boundary condition. Examples include the various invasion processes of reservoir engineering such as water flooding and gravity drainage, recrystallization of metals, melting of ice, and evaporation of droplets. Such problems as these have in recent years been considered under the name of "Stefan's Problem" [3, 5, 15].

The simplest form of Stefan's problem [5] will be treated; namely, that of finding the value of a function $u(x, t)$ in the region $0 < x < x(t)$, where $x(t)$ is the position of the moving boundary, for which

$$(1.1)\qquad \begin{aligned} &(a) && \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, && 0 < x < x(t) \\ &(b) && \frac{\partial u}{\partial x}(0, t) = -1, && t > 0 \\ &(c) && u(x(t), t) = 0, && t \geq 0 \\ &(d_1) && \frac{dx(t)}{dt} = -\frac{\partial u}{\partial x}(x(t), t), && t > 0 \\ &(e) && x(0) = 0 \end{aligned}$$

It is easy to see [5] that $(1.1d_1)$ is equivalent to

$$(1.1d_2)\qquad x(t) = t - \int_0^{x(t)} u(x, t)\, dx,$$

provided the other conditions are maintained.

Evans, Sestini, and Datzeff [3, 5, 15] have demonstrated the existence and uniqueness of the solution of (1.1) for $0 \leq t < \infty$. Landau [10] has shown that the system (1.1) is non-linear, even though (1.1a) is the classical heat flow equation. The evaluation of the solution is not yet in a definitive state. Evans, Isaacson, and MacDonald [6] have proposed a series method. The coefficients in the series are very difficult to obtain, and no radius of convergence has been given. Landau [10] has obtained some numerical results for a similar problem using an explicit difference equation, but no proof that the solution of his difference equation converges to that of the differential equation as the mesh size decreases was offered. Kolodner [9] has given an approximate treatment of the evaporation problem mentioned earlier; however, a complete treatment of his results has not appeared.

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