FIXED POINTS, FIXED SETS, AND M-RETRACTS

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Introduction. The object of this paper is to establish relations among sets in a space X, fixed points in the space $S(X)$ of closed subsets of X, embeddings of X in a cube, and embeddings of $S(X)$ in a cube.

If X has the fixed point property does $S(X)$ have? If X is homeomorphic to a retract of a cube is $S(X)$? We answer the converse of these questions in the negative. A fixed point theorem for $S(X)$ is proved, a fixed set theorem of J. L. Kelley [3] is generalized, and a new kind of embedding of X in a cube is shown (1) to imply that there exists an ordinary embedding of $S(X)$ in a cube and (2) to characterize Peano spaces. It is shown that the space of subsets, the space of subsets of the space of subsets, etc., of a retract of a cube, is a retract of a cube.

Multi-valued functions will be denoted F, G, H, and single valued functions f, g, h. Let $(D,*)$ be a directed set. Let X, Y, Z denote topological spaces and x, y, z, elements of these spaces, respectively. A point $y_0 \varepsilon Y$ is said to be in the *cofinal limit* of a sequence of sets $\{Y_d\}$ if whenever V is an open set containing y_0 there is a cofinal subset C contained in D such that $c \varepsilon C$ implies that $V \cap Y_c \neq 0$. Similarly, y_0 is an element of the *residual limit* of $\{Y_d\}$ if there is a residual subset R contained in D such that $r \varepsilon R$ implies that $V \cap Y_r \neq 0$. A multi-valued function $F: X \rightarrow Y$ is said to be *continuous* at x_0 if $\{x_d\} \rightarrow x_0$ implies that $F(x_0) =$ cofinal limit $\{F(x_d)\} =$ residual limit $\{F(x_d)\}$. $S(Y)$ denotes the set of all non-null closed subsets of Y. If U and V are open sets in Y define $N(U, V) = \{a \varepsilon S(Y) \mid A \subset U \text{ and } A \cap V \neq 0\}$. The set of all $N(U, V)$ such that U and V are open in Y will be used as a sub-basis for the open sets in $S(Y)$. This is equivalent to the topology used by Frink [1] and is equivalent to the metric topology [5] when X is a metric space. Define $S_1(X) = S(X)$ and $S_n(X) = S(S_{n-1}(X))$ for $n > 1$. The following three lemmas are proved in [6].

Lemma 1. *A point-closed multi-valued function F from a Hausdorff space X to a compact Hausdorff space Y is continuous if and only if*

1. *V open in Y and $F(x_0) \cap V \neq 0$ implies that there is an open set U containing x_0 such that $F(x) \cap V \neq 0$ whenever x is an element of U and*
2. *V open containing $F(x_0)$ implies that there exists an open set U containing x_0 such that $F(U) \subset V$.*

Lemma 2. *If $F: X \rightarrow Y$ is a continuous multi-valued function and Y is a T_1 space then $F(x)$ is closed for all x; that is F is a point-closed function.*

Lemma 3. *If $F: X \rightarrow Y$ is continuous, $\{x_d\} \rightarrow x_0$, $y_d \varepsilon F(x_d)$, and $\{y_d\} \rightarrow y_0$ then $y_0 \varepsilon F(x_0)$.*

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