THE LIE RING OF A SIMPLE ASSOCIATIVE RING

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In previous papers [1], [2] we considered the structure of the Lie ring of a simple ring A; that is, the structure of the ring derived from A by the new multiplication $[a, b] = ab - ba$ for a, b in A.

We define $[A, A]$ to be the additive subgroup of A generated by all the commutators $xy - yx$ in A. Clearly $[A, A]$ is itself a Lie ring under the new multiplication defined above for A. It has been conjectured that if A is a simple ring of characteristic different from 2, then $[A, A]$, as a Lie ring, has all its proper Lie ideals in the center of A. In [2], we proved this conjecture to be correct in the special situation where A was a division ring.

In this paper we prove the conjecture to be true for all simple rings, however, we need the added proviso that this ring have characteristic which is neither 2 nor 3. In [2], we were able to establish the desired result, even in characteristic 3, because, for a division ring, we were able to use the Brauer-Cartan-Hua theorem as a potent tool. The lack of such a theorem for arbitrary simple rings, in our method of proof at least, prevents the attainment of the final goal in case the ring has characteristic 3. We conjecture later in the body of the paper that a certain analogue of the Brauer-Cartan-Hua theorem is true for all simple rings. At any rate, up to a certain critical point, characteristic 3 does not make its presence felt, and so we are still able to obtain relevant results even in that case.

If A is an associative ring, and if U is a Lie ideal of $[A, A]$, then in [2], we considered the "associated subring", $S(U)$, of U which is defined by $S(U) = \{ x \in A \mid [x, A] \subset U\}$. We group into Lemma 1 results which we established about $S(U)$ in [2] and which will be relevant to the material of this paper. These are

Lemma 1. *If U is a Lie ideal of $[A, A]$ then*
1. *$S(U)$ is a subring of A*
2. $[U, U] \subset S(U)$
3. $[U, S(U)] \subset S(U)$.

We proceed to

Theorem 2. *If A is a simple ring of characteristic different from 2, and if U is a Lie ideal of $[A, A]$, $U \neq [A, A]$, then*

$$U^{(3)} = [[[U, U], [U, U]], [[U, U], [U, U]]] = (0).$$

Proof. $S = S(U)$ is a subring of A. Since $[U, U] \subset S$, we can assume that $S \neq (0)$, for otherwise the theorem would already be true. Let s_1, $s_2 \in S$ and

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