EQUICONVERGENCE OF CESÀRO AND RIESZ TRANSFORMS OF SERIES

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1. **Introduction.** There are striking similarities in results obtained by V. Garten [5] concerning Cesàro transforms of Tauberian series and results obtained by the author (Agnew [1], [3] and other references in [2]) for Riesz transforms of these same series. These similarities suggest that, for interesting classes of series including many divergent series, the Cesàro and Riesz transforms may be more closely related than have heretofore been suspected. Investigation of this matter reveals that there are surprisingly large classes of series having Cesàro and Riesz transforms which, whether they be convergent or divergent, are equiconvergent. By equiconvergence of the Cesàro and Riesz transforms $C_r(n)$ and $R_r(n)$ of order r of a given series $\sum u_k$, we mean that

$$\lim_{n\to\infty} [C_r(n) - R_r(n)] = 0. \tag{1.1}$$

It is assumed that r is a real number which is not necessarily an integer. We shall show in Theorem 3.1 that if $r > 0$ and $\sum u_k$ is a series for which $\lim_{k\to\infty} u_k = 0$, then (1.1) holds. In Section 4 we shall prove a theorem which implies that if $r > 0$ and $\sum u_k$ is a series having a bounded sequence of partial sums, then (1.1) holds. Finally, in Section 5, we extend these theorems to cover equiconvergence of the Cesàro transform $C_r(n)$ and the Riesz transform $R_r(\omega)$ in which ω is a real parameter not restricted to integers.

Our results on equiconvergence neither imply nor are implied by classic facts about convergence of Cesàro and Riesz transforms. The fact that the Riesz method for evaluation of series includes the Cesàro method means that if $\lim C_r(n)$ exists, then (1.1) holds. The fact that the Cesàro method does not include the Riesz method means that there is a series $\sum u_k$ for which $\lim R_r(n)$ exists and (1.1) fails to hold. It is, however, a corollary of our results that a series $\sum u_k$ for which $\lim R_r(n)$ exists and (1.1) fails to hold cannot be a series for which $\lim u_k = 0$ and cannot be a series having a bounded sequence of partial sums.

It is an immediate consequence of our results that theorems of Garten [5] on Cesàro transforms imply corresponding theorems on Riesz transforms, and that theorems on Riesz transforms imply corresponding theorems on Cesàro transforms. For a bibliography of this subject, see Agnew [2].

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