IRREGULAR BOREL MEASURES ON TOPOLOGICAL SPACES

By George Swift

1. **Introduction.** Let X be a topological space, let the Borel sets of X be the σ-algebra $\mathfrak{B}$ generated by the class $\mathfrak{C}$ of all compact sets of X, and let $\mathfrak{D}$, $\mathfrak{U}$, and $\mathfrak{V}$, denote the classes of all closed, open, and open Borel sets of X, respectively. Any measure whose domain of definition includes the Borel sets of X is said to be a Borel measure on X. If μ is a Borel measure on X, then μ is outer (inner) regular at a Borel set A if, and only if

$$(1.1)\qquad \mu(A) = \inf\{\mu(V): A \subset V \varepsilon \mathfrak{V}\} \qquad (= \sup\{\mu(C): A \supset C \varepsilon \mathfrak{C}\});$$

otherwise μ is outer (inner) irregular at A. If μ is both outer and inner regular at a Borel set A, μ is said to be regular at A; otherwise μ is irregular at A.

It is easy to see that, if X is a σ-compact Hausdorff space, if μ is a Borel measure on X, and if $A \varepsilon B$, then

$$(1.2)\qquad \mathfrak{U} = \mathfrak{V},$$

$$(1.3)\qquad \mathfrak{D} \subset \mathfrak{B},$$

and

$$(1.4)\qquad \sup\{\mu(C): A \supset C \varepsilon \mathfrak{C}\} = \sup\{\mu(D): A \supset D \varepsilon \mathfrak{D}\}.$$

Properties concerning irregular Borel measures and the sets at which they are irregular are discussed in this paper.

2. **Examples.** We here give simple examples of irregular Borel measures on topological spaces which will be referred to later in the paper.

Example 2.1. Let $X = \{x_1, x_2\}$ and let the open sets of X be exactly the sets 0, $\{x_1\}$, and X. Clearly X is a non-Hausdorff topological space. Let μ be the measure on X defined by the following relations

$$\mu(A) = \begin{cases} 1 & \text{if} \quad x_1 \varepsilon A \quad \text{and} \\ 0 & \text{otherwise} \end{cases}$$

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