A SINGULAR INTEGRAL WHOSE KERNEL INVOLVES A BESSEL FUNCTION

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1. **Introduction.** We present here two theorems on a singular integral whose kernel involves a Bessel function. The second of these theorems is applied to establish the existence and compute the value for the limit of

$$(1) \qquad \sup_{x>0} \left| \int_0^x t^{-\lambda} J_\nu(t)\, dt \right| \qquad (\nu \to \infty),$$

where λ is a constant $\geq -\frac{1}{2}$ and $J_\nu(t)$ is the Bessel function of order ν. The precise results are stated in section 2.

The expression (1), whose study motivated this paper, arose in the course of recent work by Calderón and Zygmund [3; 220, (6.1)] for which they needed to establish boundedness of (1) as $\nu \to \infty$ for fixed $\lambda \geq 1$. This was done by G. Szegö (unpublished) for $\lambda = 1$; his proof was extended to $\lambda > 1$ by Calderón and Zygmund. We have devised several other proofs of boundedness, some quite brief. A particularly simple one, valid for fixed $\lambda > -\frac{1}{2}$, is included as part of section 7, concerned with upper and lower bounds for expressions related to (1). Of course, the more precise results concerning (1), mentioned in the first paragraph, imply boundedness also for $\lambda = -\frac{1}{2}$.

The connection between the second theorem on singular integrals and (1) is found in the theorem which asserts that the areas bounded by the t-axis and the successive arches of the graph of $t^{-\lambda} \mid J_\nu(t) \mid, t \geq 0$, form a decreasing sequence for certain ranges of fixed λ and ν. This was first established by Cooke [4] for $\lambda \geq 0, \nu > -1$. For the overlapping range $\lambda \geq -\frac{1}{2}, \mid \nu \mid > \frac{1}{2}$, Makai [8] recently provided an elegant proof of this theorem, as a special case of more general results.

The first arch, which is the one with the largest area, is understood to start at $t = 0$ even when $J_\nu(0) \neq 0$, *i.e.*, even for $\nu \leq 0$. All subsequent arches always go from one zero of $J_\nu(t)$ to the next.

The Cooke-Makai theorem implies that the integral in (1), for appropriate λ and ν, is positive and attains its absolute maximum at $x = j_{\nu 1}$, the first positive zero of $J_\nu(x)$. The choice of λ, ν depends on whether Cooke's formulation or Makai's is followed. As we shall be concerned chiefly, but not exclusively, with $\nu \to \infty$, Makai's version will allow us generally, but not invariably, the broader scope, since it includes certain negative values of λ while Cooke's does not.

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