REPRESENTATIONS BY HERMITIAN FORMS IN A FINITE FIELD

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1. **Introduction.** Let $q = p^n$, $p > 2$ and suppose that

$$\theta \in GF(q^2), \qquad \theta^2 = \nu \in GF(q) \qquad \text{but} \qquad \theta \notin GF(q). \tag{1.1}$$

Then, $\alpha = a + b\theta \in GF(q^2)$ if $a, b \in GF(q)$. By the conjugate of α we mean $\bar{\alpha} = a - b\theta$. If $A = (\alpha_{ij})$ is a square matrix, $\alpha_{ij} \in GF(q^2)$, let $A^* = \bar{A}' = (\bar{\alpha}_{ij})'$ where the prime denotes transpose. Then A is said to be Hermitian if and only if $A^* = A$.

In this paper we consider the problem of determining the number $N_t(A, B)$ of $m \times t$ matrices U such that

$$U^*AU = B, \tag{1.2}$$

where A and B are Hermitian, A is non-singular of order m, B is of order t and rank $r \leq m$, and all matrices have elements in $GF(q^2)$. It is first proved (Theorem 1) that

$$N_t(A, B) = \prod_{i=1}^{r} (q^{2m-2i+1} + (-q)^{m-1})N_{t-r}(A_r\,, 0), \tag{1.3}$$

where A_r is Hermitian and non-singular of order $m - r$. The value of $N_t(A, 0)$ is given by Theorem 4. Then using this theorem and (1.3) we obtain the value of $N_t(A, B)$. Incidentally we find (Theorem 3) that the number of Hermitian matrices of order m and rank r is

$$N(m, r) = q^{\frac{1}{2}r(r-1)} \prod_{i=1}^{r} \frac{q^{2m-2(r-i)} - 1}{q^i - (-1)^i}. \tag{1.4}$$

In §7 we consider a certain sum $H(B, s)$, which for $B = 0$ reduces to $N(t, s)$. Using a relation between $N_t(A, B)$ and $H(B, s)$ we are able to explicitly evaluate the latter. In §8 we give an application of the sum $H(B; s)$ to the solution of a problem in partitions of Hermitian matrices.

The methods employed here are similar to those used in [2] and [3] in the treatment of the analogous problems for symmetric and skew-symmetric matrices, respectively. It is worth noting that all of the formulas obtained in the present paper hold as well if A and B are skew-Hermitian ($A^* = -A$, $B^* = -B$), for then the matrices θA, θB are Hermitian and may be used in place of A and B above.

2. **Notation and preliminaries.** Let $q = p^n$, $p > 2$. Numbers of $GF(q^2)$ will be denoted by lower case Greek letters $\alpha, \beta, \cdots, \xi$, except as indicated.

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