# POLYNOMIALS WHOSE ZEROS LIE ON THE UNIT CIRCLE

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**1. Introduction.** Let

$$P(z) = \prod_{j=1}^{n} (1 - z/\omega_j), \tag{1}$$

where the points $\omega_j$ lie on the unit circle $C$. It has been shown by Cohen[1] that, on some path $\Gamma$ which joins the origin to $C$, the inequality $| P | < 1$ holds everywhere except at $z = 0$. In an oral communication, C. Loewner has established the existence of a polynomial (1) for which every radius of the unit disc passes through a point at which $| P | > 1$.

We will describe (see Theorem 1) a very simple example of a polynomial (1) with the property that on each radius of the unit disc there exist two points $z'$ and $z''$ such that $| P(z') | < 1$ and $| P(z'') | > 1$.

In connection with Theorem 1, the following question might be asked: Does there exist a universal constant $L$ such that for every polynomial (1) the inequality $| P | < 1$ holds on a path which connects the origin to $C$ and has length at most $L$? This question has recently been answered in the negative by G. R. MacLane [2].

Section 3 deals with the polynomials (1) in the cases $n \leq 4$. In these cases there always exist two half-lines from the origin on which

$$| P(z) | \leq | 1 - | z |^n | \quad \text{and} \quad | P(z) | \geq 1 + | z |^n, \tag{2}$$

respectively. Here we point out the problem of determining the greatest degree $n$ for which a polynomial (1) always satisfies the inequalities (2) on two appropriate radii of the unit disc or on two half-lines from the origin.

**2. The example.** The polynomial to be described is of the form

$$P(z) = \prod_{j=1}^{q} [1 + (z/\omega_j)^j]^{k_j} \qquad (| \omega_j | = 1; j = 1, 2, \cdots, q). \tag{3}$$

Roughly speaking, each factor determines a set of directions $\theta$, of total range slightly less than $\pi$, such that on every radius in one of these directions $P(z)$ takes values of modulus greater than 1. The crucial problem in the construction is this, to choose the integers $k_j$ in such a way that each factor bears the sole responsibility, on some circular arc concentric with the unit circle, of determining the signum of log $| P(z) |$.

Let $Aj$ be the set of all $\omega$ on $C$ for which

$$-\pi/3 \leq \arg(\omega/\omega_j)^j \leq \pi/3, \qquad \text{modulo } 2\pi, \tag{4}$$

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