LOCAL PROPERTIES OF OPEN MAPPINGS

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Standing Hypothesis. Let f be an open mapping (here *mapping* is synonymous with *continuous transformation*) of a compact metric space X onto Y. Then Y is metric and the distance function ρ will be used for both X and Y. Use will be made of a countable basis $\{N_i\}_{i=1}^{\infty}$ of neighborhoods for X which have the further property that the diameter of $N_i \rightarrow 0$ as $i \rightarrow \infty$.

In this setting local topological properties have three places of incidence: in Y; in the inverses, $f^{-1}(y)$, of points $y \in Y$; and in X. The results of this paper are all of the form: if Y has a certain property and the inverses of points $y \in Y$ have another, then X has a related property.

Definition. The mapping f is *regular* at a point $x \in X$ if and only if for each neighborhood U of x, there exists a neighborhood V of x such that if $y \in f(V)$, then $f^{-1}(y) \cdot V$ lies in a single component of $f^{-1}(y) \cdot U$.

Theorem 1. *If, in addition to the standing hypothesis, for each $y \in Y$, $f^{-1}(y)$ is locally connected on a dense subset of itself, then f is regular at each point of a dense G_δ subset of X.*

Proof. Let $T = \{(i, j, k): \overline{N}_k \subset N_j,\ \overline{N}_j \subset N_i\}$. For each $t = (i, j, k) \in T$, let $Y_t = \{y: y \in f(\overline{N}_k)$ and $f^{-1}(y) \cdot N_j$ lies in a single component of $f^{-1}(y) \cdot \overline{N}_i\}$. Then for each $t \in T$, Y_t is closed. For suppose $\{y_s\}_{s=1}^{\infty}$ is a sequence of points of Y_t which converges to y, and that $t = (i, j, k)$. Then $y \in f(\overline{N}_k)$ as each $y_s \in f(\overline{N}_k)$. For each positive integer s let M_s denote the component of $f^{-1}(y_s) \cdot \overline{N}_i$ which contains $f^{-1}(y_s) \cdot N_j$. The sequence $\{M_s\}_{s=1}^{\infty}$ has a convergent subsequence whose limit, M, is a continuum, is a subset of $\overline{N}_i$, and, as f is open, contains all of $f^{-1}(y) \cdot N_j$. Therefore $y \in Y_t$.

For each $t \in T$, let i_t, j_t, k_t denote the terms of t. If U is open in X and $M > 0$, then $G_{MU} = \{Y_t : i_t > M$ and $\overline{N}_{k_t} \cdot U \neq 0\}$, covers $f(U)$. For if $y \in f(U)$, then $f^{-1}(y)$ is locally connected at some point $x \in f^{-1}(y) \cdot U$. Let N_i be one of the basis neighborhoods of x, where $i > M$. Then there exists a neighborhood N_j of x, such that $\overline{N}_j \subset N_i$ and $f^{-1}(y) \cdot N_j$ lies in a single component of $f^{-1}(y) \cdot \overline{N}_i$, and a neighborhood N_k of x, such that $\overline{N}_k \subset N_j$. Let $t = (i, j, k)$; then $t \in T$, $i_t = i > M$, and $\overline{N}_k \cdot U \neq 0$. Hence $y \in Y_t$.

For each positive integer n, the open set X_n defined by: $X_n = \bigcup \{f^{-1}(U') \cdot N_{j_t} : t \in T,\ i_t > n,\ U'$ is open in Y, and $U' \subset Y_t\}$ is dense in X. For if U is open in X, then $f(U)$ is open in Y. Therefore, as shown above, there exists a $Y_t \in G_{nU}$ which contains an open subset, U' of $f(U)$, because G_{nU} is a countable covering of $f(U)$ and Y is second category. Furthermore, as $Y_t \in G_{nU}$, $i_t > n$.

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