A DOMINATED-CONVERGENCE THEOREM

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As Daniell showed [1], it is possible to develop the theory of the Lebesgue integral so as to give the essential role to the order properties. Recently this approach has been extended [2] so as to provide a theory of order-preserving maps which specializes to various integration processes, spectral resolutions, etc.

As in Daniell's theory, one begins with an elementary integral or mapping I_0, defined on a subset E of some lattice F (in Daniell's case, a real function space) and mapping E monotonically into a partially ordered set G. One then extends the domain of definition of the mapping so that the extended map (or integral) I possesses desirable properties. Previous to introducing algebraic assumptions, the outstandingly important properties of I are lattice properties and convergence properties, the latter being asserted in a generalized form of the Lebesgue dominated convergence theorem. The mapping being defined, it is shown that algebraic properties of the elementary mapping, such as additivity or linearity, are preserved in the extended mapping I.

However, in [2] the fundamental convergence theorem departs in two respects from the program thus sketched. First, the range-space G is assumed to be an additive group, even though no algebraic properties are postulated for the mapping I_0. Thus one postulate of an algebraic nature intrudes in a theory otherwise free of algebra. Second, G is assumed to be "normal", which essentially asks that it be isomorphic with a subset of a real function space. This is stronger than the requirement that G, topologized in conformity with its order convergence, be a Hausdorff space; also, it brings in the real number system in addition to the partially ordered sets F and G.

The purpose of the present note is to establish a lattice property and a convergence property (a generalized dominated-convergence theorem) of the extended map under hypotheses weaker than those in [2] and expressible in terms of order alone. In particular, "normality" is replaced by a separation property slightly weaker than the usual Hausdorff property.

1. **Definitions.** As in [2], we shall develop a "countable" and an "unrestricted" theory together by the device of brackets; in definitions, theorems, etc., either all bracketed expressions are to be included, or else all are to be omitted.

If G is partially ordered by an antisymmetric relation $\geq$, and $S \subset G$, and $b \in G$, then b is an upper bound of S if $g \in S$ implies $g \leq b$; and b is the supremum $\vee\, S$ of S if it is an upper bound of S, and for every upper bound b' of S it is true that $b' \geq b$. Lower bounds and the infimum $\wedge\, S$ are defined dually. The set G is *Dedekind* [σ-] *complete* if for every non-empty [countable] subset

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