METHODS OF CONSTRUCTING CERTAIN STOCHASTIC MATRICES. II

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1. **Introduction.** In this paper (which is a continuation of [3]) I make use of an extension of a theorem of A. Brauer [2; Theorem 27], [3; Theorem 2] to derive further sufficient conditions for a set of real numbers to be possible characteristic roots of a stochastic matrix. The main result established is Theorem 3 in §3 below. The extension of Brauer's theorem together with the proof given in §2 below was very kindly communicated to me by Professor R. Rado; and I am grateful to him for allowing me to reproduce it here.

2. **An extension of a theorem due to A. Brauer.**

THEOREM 1. *Let A, X, Ω, C be matrices, with complex elements, of type $m \times m$, $m \times d$, $d \times d$, $d \times m$ respectively, where $1 \leq d \leq m$. If rank $X = d$ and if $AX = X\Omega$ then*

$$\det(A - XC)\cdot\det \Omega = \det A\cdot\det(\Omega - CX). \tag{1}$$

COROLLARY. *For any number λ*

$$(\lambda I_m - A)X = \lambda X - AX = \lambda X - X\Omega = X(\lambda I_d - \Omega),$$

whence in (1) A may be replaced by $\lambda I_m - A$, and Ω by $\lambda I_d - \Omega$ giving

$$\det(\lambda I_m - A - XC)\cdot\det(\lambda I_d - \Omega) = \det(\lambda I_m - A)\cdot\det(\lambda I_d - \Omega - CX). \tag{2}$$

For $d = 1$, (2) is the statement of Brauer's theorem.

Proof. The partitionings of matrices that follow are such that the operations required are possible. Now there exists a non-singular matrix $S = (XY)$.

Write

$$S^{-1} = \begin{pmatrix} U \\ V \end{pmatrix}.$$

Then

$$I_m = \begin{pmatrix} U \\ V \end{pmatrix}(XY) = \begin{pmatrix} UX & UY \\ VX & VY \end{pmatrix}.$$

Write

$$C = (C_1\ C_2), \qquad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \qquad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}.$$

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