REGULAR CURVES AND REGULAR POINTS OF FINITE ORDER

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J. R. Kline has raised the question whether or not, for any positive integer n greater than two, there exists a continuous curve such that, between each pair of its points, there are exactly n simple arcs mutually exclusive except for end points. For n equal to one and two the arc and the simple closed curve, respectively, have this property. J. H. Kusner [2] has settled this question for n equal to three or four. It is shown in Part I of this paper that for any integer n greater than two no such curve exists. The final part of this paper answers two questions raised by W. L. Ayres [1] concerning continuous curves containing points of only two orders. P. Urysohn [3] has constructed examples of curves of this type, namely, for any integer n greater than two a curve containing points of orders n and $2n - 2$ only. G. T. Whyburn [5] has shown that if a continuous curve contains points of only two orders, m and n, then m is greater than or equal to $2n - 2$. Ayres conjectured that if $m > n > 2$ then (1) the points of order m must be countable and (2) for some integer k, $m = k(n - 1)$. Examples are given to show that neither of these conjectures is true.

The problems considered in this paper were suggested to me by Professor J. R. Kline and I am indebted to him for his guidance of this work. I also wish to acknowledge Professor R. D. Anderson's guidance of the final stages of this work.

PART I

Theorem 1.1 *For n an integer greater than two there exists no locally compact continuous curve M such that each pair of points of M are the end points of exactly n arcs mutually exclusive except for end points.*

Proof. Suppose M is such a curve, then M has the following properties:

Property 1. Every pair of points can be separated by n points. This is an immediate consequence of the Theorem [6]: *If two points A and B of a locally compact continuous curve M are separated in M by no n points, then there are $n + 1$ independent arcs from A to B.*

Property 2. M is a regular curve. This is a direct consequence of the above. M is therefore hereditarily locally connected and the local separating points are everywhere dense.

Property 3. The boundary of every open set contains at least n points, thus every point is of order at least n. If p is a point of M, U an open set containing p, and q is a point which does not belong to U, then there exist n independent arcs from p to q. Since $F(U)$ separates M each arc must intersect $F(U)$, therefore $F(U)$ contains at least n points.

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