HAUSDORFF-BESICOVITCH DIMENSION OF BROWNIAN MOTION PATHS

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1. **Introduction.** The subject of this note is the Hausdorff-Besicovitch dimension of Brownian motion paths. The first progress in this connection was announced by Paul Lévy in 1951. He showed that if $f(x) = x^2 \log \log x^{-1}$, $(x > 0)$, then the Hausdorff f-measure of the n-dimensional Brownian path $x_\omega[0, t] = (x: x = x(s, \omega)$ for some $s \in [0, t])$ is almost always $\leq a(n)t$, where $a(n)$ is a positive constant, depending on n alone, and that the ordinary Hausdroff-Besicovitch dimension of the complete path $x_\omega\, [0, +\infty] = (x: x = x(s, \omega)$ for some $s \geq 0)$ is almost always 2. These and other interesting results are proved in [5].

We proceed in a somewhat different direction, comparing the Hausdorff-Besicovitch measure and dimension of a Borel set $E \subset [0, +\infty]$ with the measure and dimension of the n-dimensional Boreal set $x_\omega\,(E) = (x: x = x(s, \omega)$ for some $s \in E)$. We define the Brownian motion, Hausdorff-Besicovitch measure and dimension, and other necessary notions below.

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2. **Preparations.** The one-dimensional Brownian motion is any version of the Markov process with transition probabilities

$$P(0, x, B) = 1 \qquad x \in B$$

$$= 0 \qquad x \notin B$$

$$P(t,\ x, B) = \int_B (2\pi t)^{-\frac{1}{2}} \exp\left(-\frac{1}{2t}(x - y)^2\right) dy \qquad (t > 0),$$

which is separable relative to the class of closed sets over the rationals (see [2] for a complete discussion).

Take $n > 0$ and a positive integer $i \leq n$, let Ω_i be a copy of the space of sample functions of the one-dimensional Brownian motion just described; ω_i a particular one; $x(t, \omega_i)$ its value at time $t \geq 0$; and let PR_i be the appropriate probability measure on Ω_i. The space of sample functions of the n-dimensional Brownian motion is $\Omega = \Omega_1 \times \cdots \times \Omega_n$, and this is provided with the probability measure $PR = PR_1 \times \cdots \times PR_n$. The value of the particular sample function $\omega = (\omega_1, \cdots, \omega_n)$, $(\omega_i \in \Omega_i)$ at time $t \geq 0$ is written $x(t, \omega) = (x(t, \omega_1), \cdots, x(t, \omega_n))$. In words: the n-dimensional Brownian motion is a random vector whose entries are n independent copies of the one-dimensional Brownian motion.

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