ON THE COEFFICIENTS OF R-UNIVALENT FUNCTIONS

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An analytic function $w = f(z)$ is univalent in a region B if its inverse $z = f^{-1}(w)$ is single-valued in the part of the w-plane covered by the image B' of B. A natural generalization of the notion of univalence is obtained if the schlicht w-plane is replaced by a closed Riemann surface R. If B' lies in R and $f^{-1}(w)$ is single-valued in the subset of R covered by B', we shall say—for want of a better term—that $f(z)$ is R-univalent in B. We may also describe this situation by saying that B' is embedded in R.

The main objective of this note is to prove the following result on functions which are R-univalent in $| z | > 1$.

Theorem: *If S_R denotes the class of analytic functions $f(z)$ which map $| z | > 1$ onto a domain embedded in a given closed Riemann surface R and which have the expansion*

$$f(z) = z + a_1 z^{-1} + a_2 z^{-2} + \cdots \tag{1}$$

near $z = \infty$, then the region of variability of the coefficient a_1 within the class S_R is contained in a circle of radius 1.

Proof. Consider a simply-connected, smoothly-bounded domain D contained in R, and an Abelian integral $t(z)$ of the second kind with pure imaginary periods which has all its poles in D. We denote by C the boundary of D, and by $p(z)$ a real harmonic function which vanishes on C and is such that $p(z) - \sigma(z)$ is regular in D, where $\sigma(z) = \text{Re}\,\{t(z)\}$. If we use the symbol

$$(u, u)_D = \iint\limits_D (u_x^2 + u_y^2)\, dx\, dy,$$

we have, by Green's formula,

$$(p - \sigma, p - \sigma)_D = \int_C (p - \sigma) \frac{\partial(p - \sigma)}{\partial n}\, ds = -\int_C \sigma \frac{\partial p}{\partial n}\, ds + \int_C \sigma \frac{\partial \sigma}{\partial n}\, ds$$

$$= -\int_C \sigma \frac{\partial p}{\partial n}\, ds - (\sigma, \sigma)_{\bar{D}} ,$$

where $\bar{D}$ is the complement of D with respect to R. Hence,

$$-\int_C \sigma \frac{\partial p}{\partial n}\, ds = (p - \sigma, p - \sigma)_D + (\sigma, \sigma)_{\bar{D}} .$$

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