TRACE ON FINITE AW^*-ALGEBRAS

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1. **Introduction.** Kaplansky [3] initiated a study of W^*-algebra by its intrinsic properties, particularly the lattice structure of the set of projections. The class of algebras so characterized is termed AW^*-algebra. Dixmier [2] points out that there are AW^*-algebras which do not admit a representation as W^*-algebras. Nevertheless, under the framework of AW^*-algebra, one [4] can satisfactorily describe those algebras of type I, due largely, perhaps, to the similarity between Abelian and minimal projections. The purpose of this paper is to extend the study to algebras of type II, with particular emphasis on the existence of a trace.

Unlike W^*-algebras, where all finite algebras have a trace, we can only show its existence under the further restriction (§5) that: (P″) there exists a completely additive (c. a.) linear transformation of the algebra onto its center, being identity on the center. This is preceded by a stronger assumption that: (P) there exists a total set of (c. a.) positive functionals on the algebra. The finite algebras satisfying (P) serve as a natural link between W^*-algebras and algebras satisfying (P″). Furthermore, the transition from (P) to (P″) (when the existence of a trace is concerned) requires no effort. In §6 it is shown that a finite AW^*-algebra satisfying (P) is an AW^*-subalgebra of some W^*-algebra; the latter of course satisfies (P). The question whether or not such algebras are already weakly closed—a question raised by Kaplansky—remains open. Sections 2-4 are devoted to some preliminary results that are needed in §§5-6. In §2 it is shown that the canonical decomposition is valid in any AW^*-algebra. In §3 there is given a sufficient condition for a C^*-algebra to be a W^*-algebra. This section shows, in particular, the algebra Steen [11] considers a W^*-algebra. Section 4 is devoted to the study of commutative AW^*-algebras. We conclude this paper with an example (§7) of an AW^*-algebra of type II, having a trace and an arbitrary given center.

The terminology of this paper is essentially that of [3], with a few additions mostly from [1]. The *central carrier* $C(e)$ of a projection e is the least central projection $\geq e$. A *decomposition* of a projection e is a set of orthogonal projections whose least upper bound (LUB) is e. A *central decomposition* is a decomposition by central projections, and a *homogeneous decomposition* is one by equivalent projections. A projection e is *simple* if $C(e)$ admits a homogeneous decomposition of which e is a member.

Throughout this paper there is frequent use of the results of [3], [4] which

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