THE REPRESENTATION OF FUNCTIONS AS LAPLACE-STIELTJES INTEGRALS

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1. The purpose of this paper is to give necessary and sufficient conditions that a function defined on a convex set in the plane may be represented as a Laplace-Stieltjes integral. The result we obtain will generalize the now classical results of Hausdorff [7], S. Bernstein [3] and Widder [8], [9] for the one-dimensional case. Some of the methods used in this paper have been used by the writer [4], [5] previously in a similar connection.

Before we state our theorem let us say a word on notation. Unless stated otherwise, the lower case latin letters a, b, h, r, s, t, x, y will designate vectors in the plane, and their components by the same letters with subscripts. By $x \cdot y$ we shall mean the usual inner product of x and y and we shall write $x \leq y$ if $x_i \leq y_i$, $i = 1, 2$. If $d\alpha(t)$ designates a measure on the plane, then when we write $\int_a^b f(t)d\alpha(t)$ we shall always mean that the integral is taken over the closed interval $a \leq t \leq b$. If $M(x, y)$ is a complex valued function defined on some Cartesian product set $E \times E$ we shall write (with Aronszajn [2]) $M(x, y) \gg 0$ if for every finite set of complex numbers $\{\xi_i\}_{i=1}^n$ and points $\{x^i\}_{i=1}^n \subseteq E$, $\sum_{i,j=1}^n \xi_i \bar{\xi}_j M(x^i, x^j) \geq 0$. We shall write $M_1(x, y) \ll M_2(x, y)$ or $M_2(x, y) \gg M_1(x, y)$ if $M_2(x, y) - M_1(x, y) \gg 0$.

Suppose now that Q is a convex set in the plane which contains the origin and $Q/2$ the set of points $x \in Q$ such that $2x \in Q$. In what follows there will be no loss in generality if we suppose that Q is open. As will be seen, our methods will work for any number of dimensions.

Theorem 1. *Let $f(x)$ be a continuous function defined on Q. Necessary and sufficient conditions that there exists a unique bounded non-negative measure $d\alpha(t)$ such that $f(x) = \int_a^b e^{x \cdot t}\, d\alpha(t)$ are, for x, $y \in Q/2$:*

$$f(x + y) \gg 0 \tag{1}$$

$$a_k f(x + y) \ll \frac{\partial f(x + y)}{\partial x_k} \ll b_k f(x + y), \qquad k = 1, 2. \tag{2}$$

(If $a_k = -\infty$ or $b_k = +\infty$, the corresponding "inequality" in (2) will be considered redundant.)

2. Before we prove this theorem it will be necessary for us to recall some known facts which will be needed for the proof. A more complete discussion of these facts will be found in N. Aronszajn [1], [2] and in A. Devinatz [4], [5]. Let E be a set of points (not necessarily having any structure) and $K(x, y)$ a

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