NOTE ON VALUES OMITTED BY p-VALENT FUNCTIONS

By W. C. Royster

1. **Introduction.** Let $\mathfrak{F}$ denote the class of functions

$$w = f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n; \tag{1}$$

p a positive integer, which are regular and p-valent in $|z| < 1$, that is, $f(z)$ assumes no value w more than p times and assumes some value exactly p times for z in the unit circle. Let $D(f)$ be the Riemann domain of the transform of $|z| < 1$ by $f(z)$. $f(z)$ will be said to omit a value α, k times ($k \leq p$) if α is covered by $D(f)$ exactly $p - k$ times.

Hayman [3] proved that $f(z) \in \mathfrak{F}$ assumes every value in $|w| < 1/4^p$ exactly p times. This result is sharp as is shown by the function $f^*(z) = z^p(1+z)^{-2p}$ or a function obtained from $f^*(z)$ by suitable rotation. Biernacki [1] proved that $f(z) \in \mathfrak{F}$ assumes every value in $|w| < \frac{1}{4}$. This result is sharp as is shown by the function $f_*(z) = z^p(1 + z^p)^{-2}$ or $f_*(z)$ subjected to a suitable rotation.

Bounded functions which belong to $\mathfrak{F}$ are considered in §2. By construction of bounded functions which belong to $\mathfrak{F}$ sharp, lower bounds for omitted values are obtained. In §3 the case of values omitted k times, $1 \leq k \leq p$, is discussed.

2. **Bounded functions.** Using Hayman's result, a theorem due to Pick [5] and Nevanlinna [4] lends itself to direct generalization for functions of $\mathfrak{F}$.

Theorem 1. *Let $f(z) \in \mathfrak{F}$ omit a value α one or more times for z in $|z| < 1$ and suppose $|f(z)| < M$ in $|z| < 1$. Then*

$$|\alpha| \geq 2^{2p-1}M^2 - M - 2^pM(4^{p-1}M^2 - M)^{\frac{1}{2}} \tag{2}$$

with equality for

$$f(z) = MK\left(\frac{f^*(z)}{M}\right), \tag{3}$$

or $f(z)$ subjected to a suitable rotation, where $K(z)$ is the inverse function of $k(z) = \dfrac{z}{(1+z)^2}$.

Proof. Without loss of generality, assume $\alpha > 0$, then

$$h(z) = \frac{f(z)}{\left(1 + \frac{f(z)}{M}\right)^2} \tag{4}$$

Received January 7, 1954: revision received June 3, 1954.