BOREL AND BANACH PROPERTIES OF METHODS OF SUMMATION

By G. G. Lorentz

1. **Introduction.** A special case of the strong law of large numbers is the result of Borel according to which, for almost all x in (0, 1), the sequence of digits a_n of the dyadic expansion $x = 0, a_1a_2 \cdots a_n \cdots$ of x is C_1-summable to zero. Rademacher's functions $r_n(x)$ are connected with the a_n's by means of the formula $r_n(x) = 2a_n - 1$, hence Borel's theorem means that almost everywhere $C_1 - \lim r_n(x) = 0$. According to J. D. Hill [6], [7], a regular method of summation $A = (a_{mn})$ has the *Borel property* (or $A \varepsilon BP$) if

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn}r_n(x) = 0 \qquad \text{a.e.} \tag{1}$$

We shall say that A has the *Banach property*, if

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn}\phi_n(x) = 0 \qquad \text{a.e.} \tag{2}$$

for each normal orthogonal system $\phi_n(x)$ on (0, 1). Banach [1] proved that (2) holds for $A = C_1$ and the present author [10] that (2) holds for $A = C_\alpha$, $\alpha > \frac{1}{2}$ and does not hold for $A = C_{1/2}$. Sufficient conditions and necessary conditions for a method A to have the Borel property have been given by Hill [6], [7]. We give here conditions of a different type (connected with some simple theorems on "summability functions" of a method A, developed by the author elsewhere [12], [13], [14]), which are expecially useful in the case when the coefficient a_{mn} as a function of n shows some degree of regularity. For wide classes of methods, in particular for classes of Riesz and Abel methods, we obtain necessary and sufficient conditions.

Relation (1) can be discussed for a somewhat more general situation. We first recall some known definitions and properties of independent functions $r_n(x)$ (independent random variables in the language of Probability). Measurable sets $E_1, E_2, \cdots$ on (0, 1) are called independent, if $m \bigcap_{k=1}^{n} E'_k = \prod_{k=1}^{n} m\, E'_k$, where each E'_k is either E_k or its complement. Real measurable functions $f_1(x)$, $f_2(x), \cdots$ on (0, 1) are independent, if for any choice of measurable sets $A_1, A_2, \cdots$ of the real line, the sets $[x: f_n(x) \varepsilon A_n]$, $n = 1, 2, \cdots$ are independent. For example, Rademacher's functions $r_n(x)$ are independent. Also, if the $f_1(x)$, $f_2(x), \cdots$ are independent and $N_1, N_2, \cdots$ denote disjoint subsets of the set N of natural numbers, and if all series $g_n(x) = \sum_{k \varepsilon N_n} a_{nk}\, r_k(x)$ converge a.e., then also the functions $g_1(x), g_2(x), \cdots$ are independent.

Received December 28, 1953: revision received June 18, 1954. A considerable part of this work has been carried out while the author held a Fellowship at the 1952 Summer Institute of the Candian Mathematical Congress.