THE GENERAL LINEAR GROUP OF INFINITE FACTORS

By Richard V. Kadison

1. **Introduction.** In [3], we determined all the uniformly closed, normal subgroups of $\mathfrak{M}_g$, the group of all invertible operators in the factor $\mathfrak{M}$. Our information was complete for factors of all types with the exception of those of type II_∞. This note is devoted to supplying the missing information in the II_∞ case. We defined $\mathfrak{M}_{gf}$ to be the uniform closure in $\mathfrak{M}_g$ of the set of invertible operators which act as a scalar multiple of the identity operator on the orthogonal complement of a subspace of finite relative dimension and $\mathfrak{M}_{gf(1)}$ to be the uniform closure of the set of those operators for which this scalar is 1. We showed, in [3], that $\mathfrak{M}_{gf}$ is a closed, normal subgroup of $\mathfrak{M}_g$ (proper and non-central when $\mathfrak{M}$ is of type I_∞ or II_∞) and that $\mathfrak{M}_{gf}$ is the direct product of $\mathfrak{M}_{gf(1)}$ and the group of non-zero, complex scalars. In Lemma 6 of [3], we showed that each uniformly closed, proper, non-central, normal subgroup, $\mathcal{G}$, of $\mathfrak{M}_g$ contains $\mathfrak{M}_{gf(1)}$ (and that each normal operator in $\mathcal{G}$ lies in $\mathfrak{M}_{gf}$). We completed the determination of the subgroups $\mathcal{G}$ for factors $\mathfrak{M}$ of type I_∞, in Theorem 4 of [3], by showing that each such subgroup, $\mathcal{G}$, is the direct sum of $\mathfrak{M}_{gf(1)}$ and some closed subgroup of the scalars. It was strongly presumed that this same result holds for factors of type II_∞. However, the proof given, failed, at one point, to encompass the II_∞ case. In the following section, we shall supply a new proof which covers both the I_∞ and II_∞ cases, thereby completing the determination of all the closed normal subgroups of the general linear groups of the various types of factors.

We are indebted to I. Kaplansky for pointing out to us the advisability of taking the quotient of our factor by the unique closed, two-sided ideal in the I_∞ and II_∞ cases.

2. **The normal subgroups.** The following result will be needed for the final determination of the closed, normal subgroups. We give three proofs, the first is based on Fuglede's Theorem [1] and the second, due to I. Kaplansky, makes use of Putnam's generalization [6] of Fuglede's Theorem. The result in question does not lie as deep as Fuglede's Theorem. The third proof avoids such considerations and shows that the result is valid in a Banach algebra with a symmetric *-operation (i.e., a^*a has positive spectrum).

Lemma 1. *If $\mathfrak{A}$ is a C^*-algebra, an operator A in $\mathfrak{A}$ lies in the center of $\mathfrak{A}$ if and only if each inner transform of A, $P^{-1}AP$ (P in $\mathfrak{A}$), is a normal operator.*

Proof I. If A lies in the center of $\mathfrak{A}$ then so does A^*, so that $AA^* = A^*A$ and A is normal as is $P^{-1}AP = A$. Assume, now, that $P^{-1}AP$ is normal for each invertible operator P in $\mathfrak{A}$. Then $P^*A^*P^{*-1}$ is normal for each invertible P

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