MAXIMAL CONVEX SETS

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1. Introduction. In an arbitrary linear space L there exist maximal convex cones which exclude their vertices. We designate these cones *semispaces*. The semispaces form the minimal intersection basis for convex sets in L.

The semispaces make it possible to give a definition of a weakened form of support which may be considered final for linear spaces. Many theorems previously proved on support and separation by M. H. Stone and V. L. Klee are more readily established using the semispace representations. For extreme points the semispace concept forms a natural tool.

Dr. T. S. Motzkin is reported by Professor V. L. Klee to have defined the concept of semispaces for finite-dimensional spaces independently at about the same time we did. Unfortunately, Dr. Motzkin's results have not been accessible. It is virtually impossible to avoid duplication of work appearing only in unpublished and inaccessible lecture notes.

2. Properties of Semispaces. Let L be a linear space with origin θ. Let x be a point of L. Then if S is a maximal convex cone with vertex x but excluding x, we cay S is a *semispace at* x or *with vertex* x. If S is a semispace at x, its reflection through x is designated by S^*.

Theorem 1. *If C is a convex set excluding a point x then there exists a semispace S at x such that $S \supset C$. (Well-ordering is assumed for infinite-dimensional L.)*

Proof. We may assume $x = \theta$. Let K be the convex cone based on C with vertex θ. Then by a theorem of J. W. Ellis [2] there exists a convex cone K' with vertex θ such that, $K' \supset K$, $K' \cup (-K') = L$ and $K' \cap (-K') = \theta$. Let $S = K' \setminus \{\theta\}$ and we have the semispace required.

Corollary 1. *If K_1 and K_2 are disjoint convex cones with common excluded vertex θ, there exists a semispace S at θ such that $S \supset K_1$ and $S^* \supset K_2$.*

Proof. The set $[(K_1 \cup \{\theta\}) - (K_2 \cup \{\theta\})] \setminus \{\theta\} = K_3$, a convex cone with θ as vertex which includes K_1 and $-K_2$. By Theorem 1 there is a semispace S at θ such that $S \supset K_3 \supset K_1 \cup (-K_2)$. Moreover $S^* \supset K_2$.

Corollary 2. (Kakutani). *If A and B are disjoint convex sets then there exists a convex set H such that $L \setminus H$ is convex, $H \supset A$, and $L \setminus H$ contains B.*

Proof. Let $A - B = C$. Then $\theta \,\varepsilon\, 'C$ and C is convex. Hence, by Theorem 1, there is a semispace S at θ such that $S \supset C$. Then the intersection set of all translates of S to points of B is such a set H.

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