SUBNORMAL OPERATORS

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Introduction. In his study of dilations and extensions of operators on a Hilbert space, Halmos [4] has defined an interesting type of operator, herein called subnormal, which arises in a natural fashion from the concept of a normal operator. An operator A acting on a Hilbert space $\mathfrak{H}$ is said to be subnormal if, on some space $\mathfrak{K}$ containing $\mathfrak{H}$, there exists a normal operator B such that $Bf = Af$ for every f in $\mathfrak{H}$; then B is called a normal extension of A. Equivalently, A is subnormal on $\mathfrak{H}$, a subspace of $\mathfrak{K}$, if the normal operator B, acting on $\mathfrak{K}$, leaves $\mathfrak{H}$ invariant, and A is the restriction of B to $\mathfrak{H}$. In particular, every normal operator is subnormal; however, non-normal subnormal operators exist in sufficient abundance to justify a study of their properties, the more so, as the behavior of some subnormal operators is rather startlingly different from that of the normal operators which they yield by extension. Probably the most remarkable example of this is given in [6], in which it is shown that there is a large class of invertible subnormal operators which do not have square roots. (Every normal operator has a square root, and, in finite dimensional spaces, every invertible operator has a square root.)

The results in this paper, generally speaking, are concerned with the relationships that exist between various concepts associated with a subnormal operator and the corresponding constructs of its normal extension. In §1, for example, in the course of characterizing subnormal operators, the notion of an operator measure is introduced, which stands in approximately the same relation to a subnormal operator as a spectral measure does to a normal operator. The relations between the spectrum of a subnormal operator and that of its normal extension form the principal part of §2, while §3 is devoted to descriptions of the ring and the non-self-adjoint weakly closed algebra generated by a subnormal operator in terms of the ring generated by its normal extension.

I am happy to asknowledge my indebtedness to Professor P. R. Halmos for his constant encouragement and helpful advice in the preparation of this paper.

1. Throughout this paper, a Hilbert space is a vector space over the complex numbers, an operator is a bounded linear transformation, and a subspace is a closed linear manifold. The normal extension B, acting on $\mathfrak{K}$, of a subnormal operator A, acting on $\mathfrak{H}$, a subspace of $\mathfrak{K}$, is the minimal normal extension of A in the sense that the smallest subspace of $\mathfrak{K}$ that contains $\mathfrak{H}$ and reduces B is $\mathfrak{K}$ itself; Halmos [4] has shown that any two minimal normal extensions

Received November 25, 1953. This paper is part of the doctoral dissertation which was written while the author held an Atomic Energy Commission Predoctoral Fellowship.