QUASI-STOCHASTIC MATRICES

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1. Introduction. This paper will attempt to generalize some of the many theorems which have been proved about stochastic matrices, applying them to a more general set of matrices, which are defined below as "quasi-stochastic."

A square matrix $A = (a_{ij})$ is called "stochastic" if all the elements are non-negative and if

$$\sum_{j=1}^{n} a_{ij} = 1 \qquad (i = 1, \cdots, n). \tag{1}$$

It is well known that $z = 1$ is the root with maximum modulus for this type of matrix, and that the characteristic vector belonging to this root is the vector with components: $x_1 = x_2 = \cdots = x_n$. For convenience we set all these components equal to 1.

A square matrix $A = (a_{ij})$ is here defined as a "proper quasi-stochastic matrix" if all elements are non-negative, and (by any simultaneous permutation of rows and columns, if necessary) it can be put into a form such that:

$$\sum_{j=1}^{k} a_{ij} + p\left(\sum_{j=k+1}^{n} a_{ij}\right) = 1 \qquad (i = 1, \cdots, k),$$

(2) and

$$\sum_{j=1}^{k} a_{ij} + p\left(\sum_{j=k+1}^{n} a_{ij}\right) = p \qquad (i = k+1, \cdots, n)$$

where k is any integer such that: $1 \leq k \leq n$, and p is any positive number.

It is evident that for $p = 1$, A is stochastic, hence the set of stochastic matrices is a subset of the set of quasi-stochastic matrices.

A square matrix with arbitrary complex elements is called a "generalized stochastic matrix" if

$$\sum_{j=1}^{n} a_{ij} = S, \qquad (i = 1, \cdots, n) \tag{3}$$

where S is any complex number.

Similarly we can generalize quasi-stochastic matrices to those with arbitrary complex elements which satisfy

$$\sum_{j=1}^{k} a_{ij} + p\left(\sum_{j=k+1}^{n} a_{ij}\right) = S, \qquad (i = 1, \cdots, k),$$

(4) and

$$\sum_{j=1}^{k} a_{ij} + p\left(\sum_{j=k+1}^{n} a_{ij}\right) = pS, \qquad (i = k+1, \cdots, n),$$

Received October 16, 1953; revision received November 12, 1954.