AN EXTENSION OF THE CLASSICAL STURM-LIOUVILLE THEORY

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1. Introduction. This paper presents an explicit solution of the eigenvalue problem of a self-adjoint differential operator with a given set of self-adjoint boundary conditions in terms of the Green's function, eigenfunctions, and eigenvalues of another problem having the same operator but different boundary conditions.

As is shown in §3, the knowledge of the Green's function of this other problem implies the knowledge of a fundamental set of solutions of the differential operator. Thus, the eigenvalue problem could be treated by reducing the differential equation to a set of first or second order equations [15] and using the methods for two-point eigenvalue problems as developed by Reid [14] and extended to k-point problems by Reid [12], Denbow [6], and Mansfield [10]. The reduction of a self-adjoint differential equation and its boundary conditions to a self-adjoint system is quite complicated. Furthermore, the results so obtained would be in terms of the rank deficiency of a matrix of order $2N(k - 1)$, where $2N$ is the order of the given differential equation, and which has to be calculated for each value λ (see [10]).

The present paper gives more easily computable results under the assumption that the given problem differs only by a relatively small number R of boundary conditions from a solved eigenvalue problem for the same differential operator. We obtain all the eigenvalues with their proper multiplicities in terms of a single transcendental function which is the determinant of an $R \times R$ matrix (Theorem 1, §4).

Our theory leads to some separation theorems. The first of these (Theorem 2, §5) gives a separation criterion involving the Green's function of one of the problems and the altered boundary condition, and is new. From this we deduce criteria involving only the change of boundary conditions and arrive at a theorem (Thereom 3) which can be deduced from the results of Morse [11] for systems of second order equations or those of Reid [13] for systems of first order equations. We show that the most general separation theorems of the classical Sturm-Liouville theory [7] can be deduced from Theorem 2, as well as from the results of Morse and Reid.

Applications to the approximation of eigenvalues, particularly for the transverse vibration of a continuous beam, are given in §§6, 7.

This paper is the result of an investigation of the one-dimensional analogue of the Weinstein method for two-dimensional eigenvalue problems [21]. It was inspired by a lecture of A. Weinstein on a special problem of the type treated here [22]. The results, however, go beyond those given by either the Weinstein or the Rayleigh-Ritz method [2].

Received November 18, 1953; in revised form, May 13, 1954. This research was supported by the United States Air Force through the Office of Scientific Research.