COMPLEX BIORTHOGONALITY FOR CERTAIN SETS OF POLYNOMIALS

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1. **Introduction.** The purpose of this note is to develop a unifying theory underlying a large family of different sets of polynomials and rational functions which can be used for expansion or interpolation purposes. Given a domain B in the complex plane, with boundary C, and a set of linear functionals $\{L_n\}$ complete over $L^2(B)$, then, according to a theorem of Walsh and Davis [13], it is possible to find a set of functionals $\{L_n^*\}$, linear combinations of the L_n, and functions $\varphi_n^*(z) \in L^2(B)$, such that both

$$(1) \qquad L_n^*(\varphi_m^*) = \delta_{mn} \qquad \text{and} \qquad \iint_B \varphi_m^* \overline{\varphi}_n^* \, dA = \delta_{mn} .$$

Under certain additional assumptions, it turns out (Theorem 2) that the functions $h_n(z) = L_{n,w}^* (1/(z - w))$ satisfy the additional biorthogonality relation

$$(2) \qquad \frac{1}{2\pi i} \int_C h_n(z) \varphi_m^*(z) \, dz = \delta_{mn} .$$

For many sets $\{L_n\}$, the $\{h_n(z)\}$ are polynomials or rational functions. We are thus afforded a way, not only of deriving sets of polynomials with many useful properties assured a priori, but also of studying well-known sets of functions from a unified point of view. In particular, the orthogonality of $\{\varphi_n^*\}$ and the biorthogonality of $\{L_n\}$ with $\{\varphi_n^*\}$ provide an L^2 theory, with all the accompanying extremal, expansion, and estimation theorems, for the functions associated with any given set of functionals $\{L_n\}$.

To illustrate the applicability of the method, three different sets of functionals $\{L_n\}$ are studied in detail in Sections 3 and 4. They yield, respectively, Tscheby-sheff polynomials of the second kind, Takenaka-Walsh interpolation rationals for the unit circle, and Faber-type polynomials for an arbitrary domain B with analytic boundary. For each of these sets of functions there follows then a unified L^2 theory, rich in relations, which yields, in part, new formulas, especially for the Faber-type polynomials.

The general method admits application not only to any number of choices for the functionals $\{L_n\}$, but also to fields into which the L^2 theory generalizes readily while conformal mapping methods become more difficult. Thus, using the present methods combined with the Bergman Kernel function [2], [3] we can produce in the theory of functions of several complex variables analogues of the polynomials of Faber type and of the Walsh-Takenaka functions. In this way, the theory of the kernel function provides for the unification of results

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