UNIFORM COMPLETENESS OF SETS OF RECIPROCALS OF LINEAR FUNCTIONS. II

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1. **Introduction.** In this paper we continue the study begun in [2], which we shall hereafter denote by I, of sets K: $\{(1 + k_p x)^{-1}\}_{p=0}^{\infty}$, of reciprocals of linear functions, where $\{k_p\}_{p=0}^{\infty}$ is a sequence of complex numbers, distinct from one another and 0, none of which is a real number less than or equal to -1. We showed in I that $M(K) = F[0, 1]$ if, and only if, the series $\sum_{p=0}^{\infty} (1 - | x_p |)$ diverges, where $x_p = [(1 + k_p)^{\frac{1}{2}} - 1]/[(1 + k_p)^{\frac{1}{2}} + 1]$. We also gave two other characterizations of sets K such that $M(K) = F[0, 1]$.

In the present paper we use the notation and terminology adopted in I and give a fourth characterization, namely: in order that $M(K) = F[0, 1]$ it is necessary and sufficient that K should be *closed in* $L^2[0, 1]$, i.e. that each function in $L^2[0, 1]$ should be the limit in the mean of some sequence of K-polynomials. Moreover, we show that if K is not closed in $L^2[0, 1]$, then the closed linear manifold generated by K in $L^2[0, 1]$ is nowhere dense in $L^2[0, 1]$.

We conclude the paper with an example to show that, in a certain sense, the fundamental theorem of F. Riesz, designated as Lemma 2.1 in I, cannot be improved upon. This example shows that the equivalence of statements (i) and (iii) in Theorem 2.1 of I, is a property of the sets K not shared by every infinite sequence from $F[0, 1]$.

2. **Equivalence of $M(K) = F[0, 1]$ and $C(K) = L^2[0, 1]$.** If S is a subset of $L^2[0, 1]$, then $C(S)$ denotes the set of all functions f in $L^2[0, 1]$ such that f is the limit in the mean of some sequence $\{f_n\}_{n=1}^{\infty}$ of S-polynomials, i.e. $\int_0^1 | f_n(x) - f(x) |^2 \, dx \to 0$, as $n \to \infty$. The statement that S is closed in $L^2[0, 1]$ means that $C(S) = L^2[0, 1]$.

Theorem 2.1. *In order that $M(K) = F[0, 1]$ it is necessary and sufficient that $C(K) = L^2[0, 1]$.*

Proof. Since each function in $L^2[0, 1]$ is the limit in the mean of some sequence of functions in $F[0, 1]$, we see that $M(K) = F[0, 1]$ implies that $C(K) = L^2[0, 1]$.

Suppose $C(K) = L^2[0, 1]$, and denote by K' the set $\{(1 + (1 + k_p)x)^{-1}\}_{p=0}^{\infty}$. *The set K' is closed in $L^2[0, +\infty]$.* For, if not, then there exists a function f in $L^2[0, +\infty]$ such that $\int_0^{+\infty} | f(x) |^2 \, dx = 1$ and $\int_0^{+\infty} f(x)(1 + (1 + k_p)x)^{-1} \, dx = 0$ for $p = 0, 1, \cdots$. If $g(x) = (1 + x)f(x)$, and a is a positive number, then

$$1 = \lim_{a \to \infty} \int_0^{a/(1+a)} | g(t/(1 - t)) |^2 \, dt = \lim_{a \to \infty} \int_0^1 | g_a(t/(1 - t)) |^2 \, dt,$$

Received October 26, 1953. Presented to the American Mathematical Society, December 28, 1952.