RIEMANN'S LOCALIZATION THEOREM FOR FOURIER SERIES

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The well known localization theorem of Riemann states that for every positive δ the difference

$$R = R(x, \delta, n, f) = S_n(x, f) - \frac{1}{\pi}\int_{-\delta}^{\delta} f(x + t)t^{-1}\sin nt\, dt$$

approaches zero uniformly in x as $n \to +\infty$, where $S_n(x, f)$ is the n-th partial sum of the Fourier series for the L-integrable function f. We shall obtain here an estimate for R in terms of the integral modulus of continuity of f using what appears to be a new integral inequality of independent interest. To state these results in convenient form we define for f in $L_1(0, 2\pi)$ the quantities

$$\omega_1(h, f) = \text{Max} \int_0^{2\pi} | f(x + t) - f(x) |\, dx \qquad (| t | \leq h),$$

$$m_1(h, f) = \text{Max} \int_0^{h} | f(x + t) |\, dt \qquad (0 \leq x \leq 2\pi).$$

The relationship of these quantities may be expressed as

Theorem 1. *Unless f is equivalent to a constant function,*

$$m_1(h, f) \leq K \,||\, f \,||_1\, \omega_1(h, f),$$

where K is an absolute constant and

$$||\, f\, ||_1 = \frac{1}{2\pi}\int_0^{2\pi} | f(x) |\, dx.$$

This is applied below to prove our principal result:

Theorem 2. *For any f in $L_1(0, 2\pi)$ not equivalent to a constant*

$$R(x, \delta, n, f) \leq K\delta^{-1}[||\, f\, ||_1 + 1]\omega_1(n^{-1}, f).$$

In both theorems the exceptional case is disposed of by replacing the vanishing ω_1 by h and $1/n$ respectively.

It will be useful in proving Theorem 2 to have available the following

Lemma. *If $g(t)$ is integrable and $v(t)$ is a continuous decreasing function in $[0, \pi]$, then*

$$\left| \int_0^{\pi} g(t)v(t)\sin nt\, dt \right| \leq 2v(0)[\omega_1(\pi/n, g) + m_1(\pi/n, g)].$$

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